The Mechanical Euclid

Containing The Elements Of Mechanics And Hydrostatics Demonstrated After The Manner Of The Elements Of Geometry

Scholar

William Whewell

THE MECHANICAL EUCLID,

containing the

ELEMENTS OF MECHANICS AND HYDROSTATICS

DEMONSTRATED AFTER THE MANNER OF

THE ELEMENTS OF GEOMETRY.

WITH AN APPENDIX

containing

REMARKS ON MATHEMATICAL REASONING.

By WILLIAM WHEWELL, D.D.,

Master of trinity college, and professor of moral philosophy
in the University of Cambridge.

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Notes

The greatest possible form of ignorance is thinking that one knows what they do not know—Plato.

There is a long history of authors attempting to put some particular science into the format demonstrated by Euclid so long ago. These attempts are valid, however the ability to achieve what was wanted by the work has always been wanting simply because, no one first laid out Euclid's work by the clarity laid down by Plato in regard to the elements which has become verified by the computer today. Dialectic means binary information processing, form and matter, limit and the material difference within limits, or again, noun and verb. There are two, and only two parts of speech, two and only two, concepts to master; this mastery is not achieved by the random multiplication of words, nor of axioms, or Laws, but by ordered systems developed by a complete recognition of binary recursion. These works are of interest to the hope of their eventual achievement. And as Plato noted, the dialectician's job is to help people understand that often what they thought they knew, they do not know. The attempts of these works by their authors is, as Aristotle pointed out, exampled by his statement: *if one cannot find anyone to argue with, argue with yourself*, which is exactly what a writer does. A writer pursues dialectic, and their failures prove it just as well as success. These works allow us to see the evolution of the mind of man.

Whewell often, actually it appears to be an affectation, used *space* for *distance.* Also, he uses *S* for space, which will be changed to a *D* for distance. Corrected.

When assigning algebraic names to common grammar names, the sequences have to be identical. see p. 17 paragraph 3 original or: If the algebraic names = {*s*, *v*, *t*} then the common grammar names = {space, velocity, time}.

destroy becomes *cancels out.* One could even use the at equilibrium, but not the word destroys. If one wants to indulge in fantasies of intelligence, one could even say, "a singularity."

preponderate has be replaced with *be greater* because I have a hard time with things that not only ponderating, but pre-ponderating as it reminds me of pre-aproved which only means not approved.

Common mistake: Does the definition of anything whatsoever change in any way other than notation, between the four members of our Grammar Matrix of Common Grammar, Arithmetic, Algebra and Geometry? Absolutely not. Therefore, is any definition correctly stated in terms of a particular member of our matrix? Again, absolutely not. I will not correct these mistakes. These mistakes do add, in Biblical Metaphor, leaven to a work which is not exactly a good practice.

We are born, as Plato noted, with the ability to perform operations upon information which is independent of any particular system of grammar. One can call this, our genetic instinct. There are a great many works in our history which the author denies this in an attempt to isolate themselves as being "special."

He states that a vinculum is equitable to braces, however he mixes their use which may be a point of contention.

Dialectical Phenomenon

Preface to a Scholars Edition

"Thus, an inductive inference requires an idea from within, facts from without, and a coincidence of the two." *Mechanical Euclid* by William Whewell

The above quote is the answer to the Universal Question of Mankind. How do we pair the perceptible with the intelligible? Whewell could not comprehend the answer. Can you? This work was by an author who lived before the computer Age of Man. As such it stands as an example of the attempt by humanity to reason, that is, it is part of the pre- or proto-literate age of man.

A mind is responsible for the management of information in order to predict life supporting behavior, i.e., a mind is one of the life support systems of a life form. By now, one should be familiar with, and realize that the computer can process all information using simple binary recursion, just as intimated in the Judeo-Christian Scripture, and exampled by Plato who called it Dialectic.

I present, a concise outline of ourselves as a life-support system whose job is information processing.

What are we? In the short of it, we are called Mind. As a mind, the whole of our power is reading the environment using our bodies senses. We then process that information in order to predict life supporting behavior which is to our advantage. We then write back to the body the result of our processing to formulate our particular behavior. Behavioral Science is then the science of predictive life supporting behavior. The sum of these acts is called judgment. We can be somewhat more detailed about our job as follows.

We are one of the life support systems of a living organism called mind. A mind is an information processor. Information is actually a synonym for "thing." Our other life support systems process things physically, we as a mind process virtual things. This is called intelligibly. A mind is then a virtual information processor. Our body, working with physical things even in terms of chemistry, is vulnerable. We do the wrong thing, eat or breathe the wrong things, we are dead. Wrong behavior ranges from the insignificant to the termination of life. The mind, however, virtualizes the environment, and thus we do our work in a completely safe environment, or again, as safe as we make the body.

Our body also responds to the environment in real time. The mind, by the use of memory, is temporally independent. What our body acquires comes in and goes out continually, it is, as Plato noted, much like a very leaky pot. The mind, however, once having acquired a virtual thing, has it as long as memory permits, and it can be used recursively without ever becoming scarce.

A mind is a clean and safe room in which we can do the job we are made to do. That job is predictive behavior, which is even called prophecy. The most common name however is knowledge, a composite word for "know" and "ledger" put into a metaphor of a living book.

We read the environment with our body, process that information, and write the results back to our body to construct our own behavior, for the moment and for the future.

Since we process things, virtually, what is the virtual construct which denotes a thing?

Every thing is some relative difference constrained by limits called correlatives. These are our two, and only two parts of speech which Plato also called a noun and a verb. Every thing is composed of these two parts. We can call them the two parts of speech, noun and verb, relative and correlative, stop and go, off and on, particular and universal, and countless other name pairs. A thing is a binary construct. This binary allows us, just like a computer, to process all information using binary recursion. It also means that just like a limited mechanical computer, we have to master two, and only two, parts of speech. Just like the computer, one part of speech is a given, and the other is applied. The computer processes information called electricity as its given; however, a computer can be built using many different kinds of relative difference, from water to light. A computer then parses electricity in order to function. We are designed to parse everything. Anyone claiming the superiority of a computer is simply too stupid to know their own job. Geometrically, our binary can be represented as a simple line segment, a perceptible example of a binary which consists of the two elements of a thing which is commonly called points and line.

It is very important to understand that we can neither create nor destroy the relative difference, a given; all we can do is parse it, or again, apply limits to it. All of information hinges on, the ability to assert the limits upon a relative, and to do this recursively. This means that information processing is independent of the relative difference and the exact same operations of parsing which is applied to produce any system of grammar is identical. Therefore, every system of grammar is produced by the exact same method of parsing as every other. In short, there is no difference, as far as being better, between Common Grammar, Arithmetic, Algebra and Geometry. We depend on the whole of our Grammar Matrix. We are designed to use this Grammar Matrix with all of its members, doing their own work. Worse is the inability to perform using the Matrix.

This binary affords us exactly four different ways to effect binary recursion, which you cannot comprehend unless your mind is complex enough to comprehend metaphor. Metaphor is the ability to comprehend the same idea in the many examples. The simple I.Q. test is based on it. In order to become literate, one must be able to comprehend that the definition of a thing is the same definition for any and every thing. It is a simple binary. This result allows us to comprehend that:

Language is Universal and Intelligible, while Grammar is Particular and Perceptible.

Grammar systems themselves are again something. Their two parts are symbol sets which are absolute, or correlative, and the method of recursively applying those symbols are relative. Thus, two squared gives us our Grammar Matrix of Common Grammar, Arithmetic, Algebra and Geometry. Because each of these is a different method of effecting binary recursion, we learn to pair them in order to acquire all information afforded to us by any thing in order to effect our behavior.

If one can keep these ideas in mind, one then realizes that the foundation of our learning is not any particular field of study, but the operation of parsing applied to all information. We then learn how to apply them to any and every thing in a consistent and provable manner.

Our Grammar Matrix affords us the ability to study all trains of reasoning as binary recursion can only produce a binary result. Geometry, correctly understood, becomes the cornerstone of reasoning, as it can afford us an interactive pairing of the virtual with the perceptible that is exact and every train of reasoning is effected even independent of time for a geometric processor demonstrates that the output of processing is concurrent with the input, i.e., faster than even the claimed speed of light.

As Plato insisted that for learning anything, “Let no one ignorant of Geometry Enter Here.”

The exciting thing about this work of Whewell is that it gives just enough information to make one realize, that in order to learn how to example our givens by which to formulate and understand information processing. The examples are the lever and inclined plain. With these one can learn not only how to take what we are born with, the ability to compare things for equality and inequality, but how to turn this into what is called mathematics, the methods by which we manage information. Thus, we are given by birth the ability to compare for equality and the ability to recursively achieve this. From this foundation of binary recursion, we can learn about its power to process all information using simple binary recursion produced by simple toys, prior to what is called a formal education.

Our problem now becomes, how to use the lever and the inclined plane as toys by which to help children learn that our ability to judge equality and recursion, produces all of the so-called axioms of information processing?

Our job, as a mind, as a life support system, is the management of dialectical phenomena in order to maintain and promote life over the entire biosphere. This job is from the moment of birth to our own death, how well we pursue it, is the sum of our intelligence.

Preface to the Fourth Edition.

"The Mechanical Euclid" is a title which perhaps requires some apology, since the word "Euclid" is here used to signify, not a person, but a system of elementary propositions, connected and demonstrated with a rigor like that of the Elements of Geometry. The work was undertaken from a conviction that, if it could be properly executed, the sciences of Mechanics and Hydrostatics might be employed, as well as Geometry, in that discipline of the mind which is an essential part of a sound education, and of which rigorous mathematical reasoning is so important and valuable an instrument. And since the University of Cambridge has recently declared itself of this opinion, by appointing the elementary portions of Mechanics and Hydrostatics as a necessary part of the ordinary examinations for degrees, the work has been carefully adapted to the scheme thus laid down by authority.

In an elementary science thus intended to be employed as a discipline of the intellect, it is desirable that the matter to be studied should be reduced to certain and fixed Propositions, as is done in Geometry. I have therefore, in this Edition, adopted the list of Propositions in Mechanics and Hydrostatics, required by the University in the examination above mentioned; and have preserved the numbers of that list without change; marking the few additional Propositions which I have introduced with the letters of the alphabet, as is done by Simson in his Euclid. When the existing scheme of University Examination has been continued a few years longer, it may be hoped that this list of Elementary Propositions in Mechanics and Hydrostatics will become classical, as the Propositions of Euclid's Elements are: so that "the eighth Proposition of Mechanics," or "the sixth of Hydrostatics," may be expressions as familiarly understood ,as "the forty-seventh of Euclid's First Book," or "the fourth of his Second."

So far as I have learnt, the Examination in the Elements of Mechanics and Hydrostatics thus appointed by be University, has, in its operation, shown a highly satisfactory prospect of the beneficial effects which it is likely to produce when its course shall have been well determined by practice. Perhaps, I may be allowed to make here one or two remarks bearing upon this subject.

One ground on which some persons may perhaps for a moment doubt the efficacy of this examination as an intellectual discipline, is this:—that the list Propositions being thus limited and known beforehand, there seems to be nothing to prevent the student from learning the demonstrations by rote, and delivering them to the examiner without understanding them. And to this I reply, that the same argument might be urged, with at least equal force, against the value of Euclid's Geometry as a part of our examinations; and yet I believe everyone practically acquainted with University and College examinations and their effects, will agree with me that Euclid's Geometry is the most effective and the most valuable portion of our mathematical education. If the examinations on Mechanics and Hydrostatics be assimilated as much as possible to the examinations in Euclid, they will have the same kind of effect, as a discipline of strict reasoning; and the study of these additional sciences will bring with it additional advantages,

arising from the more extensive and varied nature of the subjects thus presented to the student's mind.

In introducing these additional sciences into the study for the usual degree, the portion which Algebra occupied in the examinations was rather diminished than increased. So far as this change was requisite to facilitate the introduction of the new portions of the examination, it will not, I think, be deemed an evil by anyone who wishes the studies of the University to be so selected and arranged as to be an intellectual discipline. For the knowledge of Algebra which is generally acquired by those who study that subject merely with a view to the ordinary degree, must be so scanty as to be of small value for the purpose just mentioned; especially when we take into account the very imperfect acquaintance with Arithmetic which students in general, according to the present practice of many places of previous education, bring to the University. Even in the hands of those who are able to use it with facility and certainty, as a language and an instrument, the great charm of Algebra is that it expresses reasonings, and obtains the result of them, without the exercise of the reason: and when students are required to follow in a general form relations and combinations of numbers which they cannot deal with in particular cases, their apprehension of the meaning and grounds of the processes must be so obscure, as to prevent the mind receiving any portion of the salutary effect which a complete mastery of the science might produce.

There are indeed a few simple algebraical terms and operations which occur so familiarly in mathematical reasonings, that the student cannot conveniently remain ignorant of them; and accordingly, the University has directed that the examination above mentioned shall include questions of this kind. These parts of algebra, extracted from Dr. Wood's Algebra by permission of the author, are given in the Introduction to the present work, along with a few other portions of Pure Mathematics, to which it is convenient to be able to refer in a succeeding part of the book.

Some of the enunciations of theorems contained in the Schedule sanctioned by the University, (in consequence, I conceive, of the wish felt by the framers of the plan that the document should be as brief as possible,) contain Propositions each of which may conveniently be separated into two or more; for instance, Proposition 8, and 16, of the Mechanics; and Proposition 1, 2, 5, 6, 10, of the Hydrostatics. Perhaps it might be convenient, when these Propositions are required in an examination, to state which *Case* is intended.

In order that the present little work may serve as guide to the student in preparing for the examination to which I have referred, I have inserted in an Appendix the Grace of Feb. 22, 1837, (by which this part of the examination was founded,) as modified by the Grace of May 11, 1842.

Trin. Coll.

March 13, 1843

Preface to the fifth edition.

In this edition I have restored to their places the *Third Book of Mechanics*, which contain the Laws of Motion, and the *Remarks on Mathematical Reasoning*, and on the *Logic of Induction*; both which portions were, with a view to brevity, omitted in the fourth edition; but are, it appears, desired by many readers. I have also inserted the Questions on Mechanics and Hydrostatics proposed in the Examinations for the present year; and the modifications of the Regulations respecting the examinations which were introduced by the Grace of March 20, 1846.

It has been noticed to me that the demonstration of Proposition 8. Book 1, may be somewhat simplified in this manner.

After the words "and therefore *DAq* is a straight line," go on thus:

And therefore, *DA* is parallel to *Cp*. Also since *CAr* is a straight line, *CA* is parallel to *Dp*. Hence *DC* is a parallelogram, and therefore *CA* = *Dp*. But since *pr* also is a parallelogram, *Dp* = *Ar*; therefore *CA* = *Ar*.

Trin. Coll

March 23, 1849.

Contents

∵The parts of Algebra required by the University in the ordinary Examination for the B. A. Degree are, in the following pages, marked with *, an asterisk.

Introduction
Elementary Pure Mathematics.

Algebra.

***(1).To define and explain Algebraical Signs.**

Article 1. The method of representing the relation of abstract quantities by letters and characters, which are made the signs of such quantities and their relations, is called Algebra.

Known or determined quantities are usually represented by the first letter of the alphabet a, b, c, d, etc. and unknown or undetermined quantities by the last y, x, w, etc.

The following signs are made use of to express the relations which the quantities bear to each other.

2. + Plus, signifies that the quantity to which it is prefixed must be added. Thus a + b signifies that the quantity represented by b is to be added to the quantity represented by a; if a represent 5, and b, 7, then a + b represents 12.

If no sign be placed before a quantity, the sign + is understood. Thus, a signifies $+ a$. Such quantities are called positive quantities.

3. − Minus, signifies that the quantity to which it is prefixed must be subtracted. Thus $a - b$ signifies that b must be taken from a; if a be 7, and b, 5, $a - b$ expresses 7 diminished by 5, or 2.

Quantities to which the sign − is prefixed are called negative quantities.

4. × *Multiplied by*, signifies that the quantities between which it stands are to be multiplied together. Thus, $a \times b$ signifies that the quantity represented by a is to be multiplied by the quantity represented by b.*

This sign is frequently omitted; thus, $a\ b\ c$ signifies $a \times b \times c$, or a full point is used instead of it; thus $1 \times 2 \times 3$, and $1 \cdot 2 \cdot 3$ signify the same thing.

5. If in multiplication the same quantity be repeated any number of times, the product is usually expressed by placing above the quantity the number which represents how often it is repeated; thus a, a × a, a × a × a, a × a × a × a, and a^1, a^2, a^3, a^4, have respectively the same signification. These quantities are called *powers*; thus a^1, is called the *first power* of a; a^2, the *second power*, or *square* of a; a^3, the *third power*, or *cube* of a; a^4, the *fourth power*, or *biquadrate* of a. The succeeding powers have no names in common use except those which are expressed by means of number; thus, a^7 is the *seventh power* of a, or a to the seventh power; and a^n is a to the n^{th} power.

The numbers 1, 2, 3, etc. are called the indices of a; or *exponents* of the powers of a.

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* By quantities, we understand such magnitudes as can be represented by numbers; we may therefore without impropriety speak of the multiplication, division, etc. of quantities by each other.

6. ÷ Divided by, signifies that the former of the quantities between which it is placed is to be divided by the latter. Thus $a \div b$ signifies that the quantity a is to be divided by b.

The division of one quantity by another is frequently represented by placing the dividend over the divisor with a line between them, in which case the expression is called a *fraction*. Thus, $\frac{a}{b}$ signifies a divided by b; and a is the *numerator* and b the *denominator* of the fraction; also $\frac{a+b+c}{e+f+g}$ signifies that a, b, and c added together, are to be divided by e, f, and g added together.

7. A quantity in the denominator of a fraction is also expressed by placing it in the numerator, and prefixing the negative sign to its index; thus a^{-1}, a^{-2}, a^{-3}, a^{-n} signify $\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^n}$ respectively; these are called the negative powers of a.

8. The *reciprocal* of a fraction is the fraction inverted. Thus $\frac{a}{b}$ is the reciprocal of $\frac{b}{a}$ and $\frac{1}{a}$ is the reciprocal of a.

9. A line drawn over several quantities signifies that they are to be taken collectively, and it is called a *vinculum*. Thus $\overline{a - b + c} \times \overline{d - e}$ signifies that the quantity represented by $a - b + c$ is to be multiplied by the quantity represented by $d - e$. Let a stand for 6; b, 5; c, 4; d, 3; and e, 1; then $a - b + c$ is $6 - 5 + 4$, or 5; and $d - e$ is $3 - 1$, or 2; therefore $\overline{a - b + c} \times \overline{d - e}$ is 5×2, or 10. $\overline{ab - cd} \times \overline{ab - cd}$ or $\overline{ab - cd}\,\big|^2$ signifies that the quantity represented by $ab - cd$ is to be multiplied by itself.

Instead of a *vinculum*, *brackets* are sometimes use as $(ab - cd)^2$, $\{a - b + c\} \cdot \{d - e\}$.

10. = *Equal to*, signifies that the quantities between which it is placed are equal to each other, thus $ax - by = cd + ad$, signifies that the quantity $ax - by$ is equal to the quantity $cd + ad$.

11. The *square root* of any proposed quantity is that quantity whose square, or second power, gives the proposed quantity. The cube root, is that quantity whose cube gives the proposed quantity, etc.

The signs $\sqrt{\ }$, or $\sqrt[2]{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, etc. are used to express the square, cube, biquadrate, etc. roots of the quantities before which they are placed.

$$\sqrt{a^2} = a,\ \sqrt[3]{a^3} = a,\ \sqrt[4]{a^4} = a, \text{ etc.}$$

These roots are all represented by the fractions ½, ⅓, ¼, etc. placed a little above the quantities, to the right. Thus a^{n}, $a^{1/3}$, $a^{1/4}$, $a^{1/n}$, represent the square, cube, fourth and n^{th} root of a, respectively; $a^{5/2}$, $a^{7/3}$, $a^{3/5}$, represent the square root of the fifth power, the cube root of the seventh power, the fifth root of the cube of a.

12. If these roots cannot be exactly determined, the quantities are called *irrational* or *surds*.

13. Points are made use of to denote proportion, thus $a : b :: c : d$, signifies that a bears the same proportion to b that c bears to d.

14. The number prefixed to any quantity, and which shows how often it is to be taken, is called its *coefficient*. Thus, in the quantities 7 ax, 6 by, and 3 dz, 7, 6, and 3 are called the coefficients of ax, by, and dz respectively.

When no number is prefixed, the quantity is to be taken once, or the coefficient 1 is understood.

These numbers are sometimes represented by letters, which are called coefficients.

15. Similar, or *like* algebraical quantities are such as differ only in their coefficients; 4 a, 6 ab, 9 a^2, 3 a^2bc, are respectively similar to 15 a, 3 ab, 12 a^2, 15 a^2bc, etc.

Unlike quantities are different combinations of letters; thus, ab, a^2b, ab^2, abc, etc, are unlike.

16. A quantity is said to be a *multiple* of another, when it contains it a certain number of times exactly: thus 16 a is a multiple of 4 a, as it contains it exactly four times.

17. A quantity is called a *measure* of another, when the former is contained in the latter a certain number of times exactly; thus, 4 a is a measure of 16 a.

18. When two numbers have no common measure but unity, they are said to be *prime* to each other.

19. A *simple* algebraical quantity is one which consists of a single term, as a^2bc.

20. A *binomial* is a quantity consisting of two terms, as $a + b$, or $2\,a - 3\,bx$. A *trinomial* is a quantity consisting of three terms, as $2a + bd + 3c$.

21. The following examples will serve to illustrate the method of representing quantities algebraically:—

Let $a = 8$, $b = 7$, $c = 6$, $d = 5$ and $e = 1$; then

$$3a - 2b + 4c - e = 24 - 14 + 24 - 1 = 33.$$

$$ab + ce - bd = 56 + 6 - 35 = 27.$$

$$\frac{a + b}{c - e} + \frac{3b - 2c}{a - d} = \frac{8 + 7}{6 - 1} + \frac{21 - 12}{8 - 5} = \frac{15}{5} + \frac{9}{3} = 6.$$

$$d^2 \times \overline{a - c} - 3ce^2 + d^3 = 25 \times 2 - 18 + 125 = 50 - 18 + 125 = 157.$$

***(2). To Add and Subtract simple Algebraical Quantities.**

22. The addition of algebraical quantities is performed by connecting those that are *unlike* with their proper signs, and collecting those that are *similar* into one sum.

Examples:

$$\begin{array}{lr} & \text{Add} \\ & 4x \\ & 3x \\ & 7a \\ & -2a \\ \hline \text{Sum} & 7x + 5a \end{array} \qquad \begin{array}{lr} & \text{Add} \\ & 5ax \\ & -ax \\ & by \\ & -cy \\ \hline \text{Sum} & 4ax + by - cy \end{array}$$

$$\begin{array}{lr} & a + 2bx - y^2 \\ & b - bx + 3y^2 \\ \hline \text{Sum} & a + b + bx + 2y^2 \end{array} \qquad \begin{array}{lr} & a + 3b \\ & a + n - 4b \\ \hline \text{Sum} & 2a + n - b \end{array}$$

23. Subtraction, or the taking away of one quantity from another, is performed by changing the sign of the quantity to be subtracted, and then adding it to the other by the rules laid down in Article 22.

From $7x$

Subtract x

Diff. $7x - x$ or $6x$

From $7x + 3a$

Subtract $5a - x$

Diff. $7x + x + 5a - 3a$ or $8x + 2a$

From $4x^2 + 5\ ax - y^2$

Subtract $3x^2 - 3ax + y^2$

Diff. $x^2 + 3\ ax - 2y^2$

***(3). To multiply simple Algebraical Quantities.**

24. The multiplication of simple algebraical quantities must be represented according to the notation pointed out in Article 4 and 5. Thus, $a \times b$, or ab, represents the product of a multiplied by b; abc, the product of the three quantities a, b, and c:

It is also indifferent in what order they are placed, $a \times b$ and $b \times a$ being equal.

25. If the quantities to be multiplied have coefficients, these must be multiplied together as in common arithmetic; the literal product being determined by the preceding rules.

Thus, $3a \times 5b = 15\ ab$; because

$$3 \times a \times 5 \times b = 3 \times 5 \times a \times b = 15ab.$$

26. The powers of the same quantity are multiplied together by adding the indices: thus, $a^2 \times a^3 = a^5$; for $aa \times aaa = aaaaa$. In the same manner,

$$a^m \times an = a^{m+n}; \text{ and } 3a^2x^3 \times 5\ axy^2 = 15a^3x^4y^2.$$

27. If the *multiplier* or *multiplicand* consist of several terms, each term of the latter must be multiplied by every term of the former, and the sum of all the products taken, for the whole product of the two quantities.

***(4). To divide simple Algebraical Quantities.**

28. To divide one quantity by another, is to determine how often the latter is contained in the former, or what quantity multiplied by the latter will produce the former.

Thus, to divide ab by a is to determine how often a must be taken to make up ab; that is, what quantity multiplied by a will give ab; which we know is b. From this consideration are derived all the rules for the division of algebraical quantities.

If only a part of the product which forms the *divisor* be contained in the *dividend*, the division must be represented according to the direction in Article 6, and the quantities contained both in the divisor and dividend expunged.

Thus $15a^2b^2c$ divided by $3a^2bc$ is $\dfrac{15a^2b^2c}{3a^2bc}$, which is equal to $\dfrac{5bc}{x}$, expunging from the dividend and from the divisor the quantities 3, a^2, and b.

***(5). To reduce Fractions to others of equal value which have a common denominator.**

29. Fractions are changed to others of equal value with a common denominator, by multiplying each numerator by every denominator except its own, for the new numerator; and all the denominators together for the common denominator.

Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$; be the proposed fractions; then $\frac{adf}{bdf}, \frac{cbf}{bdf}, \frac{edb}{bdf}$, are fractions of the same value with the former, have the common denominator *bdf*. For $\frac{adf}{bdf} = \frac{a}{b}$; $\frac{cbf}{bdf} = \frac{c}{d}$ and $\frac{edb}{bdf} = \frac{e}{f}$ (Art, 28); the numerator and denominator of each fraction having been multiplied by the same quantity viz.—the product of the denominators of all the other fractions.

30. When the denominators of the proposed fractions are not prime to each other, find their greatest common measure; multiply both the numerator and denominator of each fraction by the denominators of all the rest, divided respectively by their greatest common measure; and the fractions will be reduced to a common denominator in lower terms than they would have been by proceeding according to the former rule.

Thus $\frac{a}{mx}, \frac{b}{my}, \frac{c}{mz}$ reduced to a common denominator are $\frac{ayz}{mwyz}; \frac{bxz}{mwyz}; \frac{cxy}{mwyz}$.

***(6). To add together simple Algebraical Fractions.**

31. If the fractions to be added have a common denominator their sum is found by adding the numerators together and retaining the common denominator. Thus,

$$\frac{2a}{5} + \frac{a}{5} + \frac{3a}{5}$$

$$\frac{a + 2x}{3} + \frac{a - x}{3} = \frac{2a}{3}$$

$$\frac{7x + y}{a} + \frac{2y - 5x}{a} = \frac{2x - 4y}{a}.$$

32. If the fractions have not a common denominator, they must be transformed to others of the same value which have a common denominator, (by Article 29), and then the addition may take place as before. Thus,

$$\frac{a}{3} + \frac{a}{5} = \frac{5a}{15} + \frac{3a}{15} = \frac{8a}{15}$$

$$\frac{a}{b} + \frac{a}{x} = \frac{ax}{6x} + \frac{ab}{6x} = \frac{ax + ab}{6x}$$

To obtain them in the lowest terms, each must be reduced to another of equal value, with the denominator which as the least common multiple of all the denominators.

$$\frac{a}{b} + 1 = \frac{a}{b} + \frac{b}{b} = \frac{a + b}{b};$$

$$2 - \frac{a}{3x} = \frac{6x - a}{3x}.$$

***(7). To multiply simple Algebraical Fractions.**

33. To multiply a fraction by any quantity, multiply the numerator by that quantity and retain the denominator.

Thus $\frac{a}{b} \times c = \frac{ac}{b}$. For if the quantity to be divided be c times as great as before, and the divisor the same, the quotient must be c times as great.

34. The product of two fractions is found by multiplying the numerators together for a new numerator, and the denominators for a new denominator.

Let $\frac{a}{b}$ and $\frac{b}{c}$ be the two fractions: then $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. For if $\frac{a}{b} = x$, and $\frac{c}{d} = y$, by multiplying the equal quantities $\frac{a}{b}$ and x by b, $a = bx$ (Article 28), in the same manner $c = dy$; therefore, by the same axiom, $ac = bdxy$ dividing these equal quantities, ac and $bdxy$ by bd, we have $\frac{ac}{bd} = xy = \frac{a}{b} \times \frac{c}{d}$.

***(8). To divide simple Algebraical Fractions.**

35. To divide a fraction by any quantity, multiply the denominator by that quantity, and retain the numerator.

The fraction $\frac{a}{b}$ divided by c, is $\frac{a}{bc}$. Because $\frac{a}{b} = \frac{ac}{bc}$, and a c^{th} part of this is $\frac{a}{bc}$; the quantity to be divided being a c^{th} part of what it tfas before, and the divisor the same.

36. To divide a quantity by any fraction, multiply the quantity by the reciprocal of the fraction. (Article 8).

If we divide c by $\frac{a}{b}$ we obtain $\frac{bc}{a}$.

For if $c \div \frac{a}{b} = x$, $c = x \times \frac{a}{b}$, or $c = \frac{ax}{b}$, and $x = \frac{bc}{a}$.

***(9). Algebraical definition of Proportion.**

37. Four quantities are said to be proportionals, when the first is the same multiple, part, or parts of the second, that the third is of the fourth.

Thus, the four quantities 8, 12, 6, 9, are proportionals; for 8 is ⅔ of 12, and 6 is ⅔ of 9.

In this case 8/12 = 6/9 and generally *a*, *b*, *c*, *d* are proportionals if $\frac{a}{b} = \frac{c}{d}$. This is usually expressed by saying *a* is to *b*, as *c* to *d*; and thus represented, $a : b :: c : d$.

The terms *a* and *d* are called the extremes, and *b* and *c* the *means*.

The fraction $\frac{a}{b}$ is called the ratio of *a* to *b*.

***(10). Algebraical consequences of Proportion.**

38. When $\frac{a}{b} = \frac{c}{d}$, if a be equal to b, c is equal to d, and if a be less than b, c is less than d, and if a be greater than b, c is greater than d.

39. When four quantities are proportionals, the product of the extremes is equal to the product of the means.

Let a, b, c, d be the four quantities; then, since they are proportionals, $\frac{a}{b} = \frac{c}{d}$; and by multiplying both sides by bd, $ad = bc$.

Any three terms in a proportion $a : b :: c : d$ being given, the fourth may be determined from the equation $ad = bc$.

40. If the first be to the second as the second to the third, the product of the extremes is equal to the square of the mean.

For (Article 39) if $a : x :: x : b$, $ab = x^2$.

41. If the product of two quantities be equal to the product of two others, the four are proportionals, making the terms of one product the means, and the terms of the other the extremes.

Let $xy = ab$, then dividing by ay, $\frac{x}{a} = \frac{b}{y}$,

$$\text{or, } x : a :: b : y.$$

42. If $a : b :: c : d$, and $c : d :: e : f$, then

$$\text{will } a : b :: e : f.$$

Because $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$, therefore $\frac{a}{b} = \frac{e}{f}$; or

$$a : b :: e : f$$

43. If four quantities be proportionals, they are also proportionals when taken *inversely*.

If $a : b :: c : d$, then $b : a :: d : c$. For $\frac{a}{b} = \frac{c}{d}$, and dividing unity by each of these equal quantities; or taking their reciprocals, $\frac{b}{a} = \frac{d}{c}$; (Article 36)

$$\text{that is, } b : a :: d : c.,$$

44. If four quantities be proportionals, they are proportionals when taken alternately.

If $a : b :: c : d$, then $a : c :: b : d$.

Because the quantities are proportionals, $\frac{a}{b} = \frac{c}{d}$; and multiplying by $\frac{b}{c}$, $\frac{a}{c} = \frac{b}{d}$, or $a : c :: b : d$.

45. Unless the four quantities are of the same kind, the alternation cannot take place, because this operation supposes the first to be some multiple, part, or parts, of the third.

One line may have to another line the same ratio that one weight has to another weight, but a line has no relation in respect of magnitude to a weight. In cases of this kind, if the four quantities be represented by numbers or other quantities which are similar, the alternation may take place, and the conclusions drawn from it will be just.

46. If $a : b :: c : d$, then *componendo*,

$$a + b : b :: c + d : d.$$

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} + 1 = \frac{c}{d} + 1$;

$$\text{therefore } \frac{a+b}{b} = \frac{c+d}{d};$$

$$\text{therefore } a + b : b :: c + d : d.$$

47. Also *dividendo*, $a - b : b :: c - d : d$.

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} - 1 = \frac{c}{d} - 1$;

$$\text{therefore } \frac{a-b}{b} = \frac{c-d}{d};$$

$$\text{therefore } a - b : b :: c - d : d.$$

48. Also *convertendo*, $a : a - b :: c : c - d$.

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{b}{a} = \frac{d}{c}$;

$$\text{therefore } 1 - \frac{b}{a} = 1 - \frac{d}{c};$$

$$\text{therefore } \frac{a-b}{a} = \frac{c-d}{c}; \text{ therefore } a - b : a :: c - d : c;$$

and by Article 43, $a : a - b :: c : c - d$.

49. If we have any number of sets of proportionals, and if the corresponding terms be multiplied together, the products are proportionals.

If $a : b :: c : d$, and $p : q :: r : s$.

$$\text{and } u : v :: x : y,$$

$$\text{then } apu : bqv :: crx : dsy.$$

For $\frac{a}{b} = \frac{c}{d}$ and $\frac{p}{q} = \frac{r}{s}$ and $\frac{u}{v} = \frac{x}{y}$;

and multiplying together equals $\frac{apu}{bqv} = \frac{crx}{dsy}$;

$$\text{therefore } apu : bqv :: crx : dsy.$$

50. If the same quantities occur in the antecedents of one set of proportionals and the consequents of another set, the resulting proportionals will be reduced. v

If $a : b :: c : d$, and $b : e :: d : f$,

$$\text{then } a : e :: c : f.$$

For $\frac{a}{b} = \frac{c}{d}$ and $\frac{b}{e} = \frac{d}{f}$; therefore $\frac{ab}{be} = \frac{cd}{df}$

and $\frac{a}{e} = \frac{c}{f}$; wherefore $a : e :: c : f$.

If $a : b :: x : y$, and $b : c :: z : x$,

and $c : d :: z : t$,

then $a : d :: x^2 : ty$.

For $\frac{a}{b} = \frac{x}{y}$, and $\frac{b}{c} = \frac{z}{x}$, and $\frac{c}{d} = \frac{z}{t}$;

$$\text{therefore } \frac{abc}{bcd} = \frac{xzz}{yxt};$$

and expunging common factors in the numerators and denominators,

$$\frac{a}{d} = \frac{z^2}{yt}$$

***(11). Of Variation.**

51. Quantities of the same kind assume different values under constant conditions, and when these different values are compared, the quantities are spoken of as *variable*, and the proportion of the different values may be expressed by two terms of a proportion instead of four.

Thus, if a man travels with a constant velocity (for example 4 miles an hour,) the distance travelled over in any one time is to the distance travelled over in any other time as the first time is to the second time; and this may be expressed by saying that the distance varies as the time, or is as the time.

52. One quantity is said to vary directly as another when the two quantities depend wholly upon each other, in such a manner that if the one be changed the other is changed in the same proportion.

If the altitude of a triangle be invariable, the area varies as the base. For if the base be increased or diminished in any proportion, the area is increased or diminished in the same proportion, (Euclid vi. 1.)

53. One quantity is said to vary *inversely* as another, when the former cannot be changed in any manner, but the reciprocal of the latter is changed in the same manner.

If the area of a triangle be given the base varies as the perpendicular altitude.

If A, a represent the altitudes, B, b the bases of two triangles, since a triangle is half the rectangle on the same base, and of the same altitude, and the triangles are equal, $\frac{1}{2}AB = \frac{1}{2}ab$, (See Geometry.)

Therefore

$$A : a :: b : B, \text{ or } A : a :: \frac{1}{B} : \frac{1}{b}.$$

54. One quantity is said to *vary as others jointly*, if, when the former is changed in any manner, the product of the others is changed in the same proportion.

The area of a triangle varies as its altitude and base jointly.

Let A, B, a, b be the altitudes and bases of two triangles as before, and S,s the areas; then

$S = \frac{1}{2}AB$, $s = \frac{1}{2}ab$ and $S : s :: AB : ab$.

55. In the same manner $A : a :: \frac{S}{B} : \frac{s}{b}$; and A varies as S directly and B inversely.

56. The symbol $\propto$ is often used for variation. Thus, the above variations may be expressed

$$A \propto \frac{1}{B},\ S \propto AB,\ A \propto \frac{S}{B}.$$

57. When the increase or decrease of one quantity depends upon the increase or decrease of two others, and it appears that if either of these latter be constant, the first varies as the other, when they both vary, the first varies as their product.

Thus, if V be the velocity of a body moving uniformly, T the time of motion, and D the distance described; if T be constant $D \propto V$; if V be constant $D \propto T$; but if neither be constant $D \propto TV$.

Let d, v, t be any other distance, velocity, and time; and let X be the distance described with the velocity, v in the time T: then

$D : X :: V : v$, because T is the same in both,

$X : d :: T : t$, because v is the same in both.

Therefore (Article 50)

$$D : d :: TV : tv; \text{ that is, } D \propto TV.$$

(12). Of Arithmetical Progression.

58. Quantities are said to be in arithmetical progression, when they increase or decrease by a common difference.

Thus 1, 3, 5, 7, 9, etc., where the increase is by the difference 2;

a, $a + b$, $a + 2b$, $a + 3b$, etc., where the increase is by the difference b;

$9a + 7a$, $8a + 6x$, $7a + 5x$, etc., where the decrease is by the difference $a + x$;

are in arithmetical progression.

59. To find any term of an arithmetical progression, multiply the difference by the number of the term minus one, and add the product to the first term, if the progression be an increasing one, or subtract the product, if a decreasing one.

Thus the 10th term of 1, 3, 5, etc. is $1 + 9 \times 2 = 19$.

The nth term of a, $a + b$, $a + 2b$, etc. is $a + \overline{n - 1b}$.

The 6th term of $9a + 7x$, $8a + 6x$, etc. is

$$9a + 7x - 5\,(a + x) = 9 + 7x - 5a - 5x = 4a + 2x.$$

60. To find the sum of an arithmetical progression, multiply the sum of the first and last terms by half the number of terms.

Thus, the sum of 10 terms of 1, 3, 5, etc. is

$$(1 + 19) \times 5 = 100.$$

For if $1 + 3 + 5 +$ etc., to 19 (10 terms) $= s$,

$19 + 17 + 15 +$ etc., to 1 (10 terms) $= s$;

therefore $20 + 20 + 20 +$ etc., to 20 (10 terms) $= 2s$,

or $20 \times 10 = 2s$, or $20 \times 5 = s$.

Also, n terms of a, $a + b$, $a + 2b$, etc.

$$= (2a + \overline{n - 1}\,b)\,\frac{n}{2}.$$

For if $a + (a + b) + (a + 2b) +$ etc.

to $a + \overline{n - 1}\,b$ (n terms) $= s$.

$$(a + \overline{n - 1}\,b)\ 4\text{-}\ (a + \overline{n - 2}\,b) + \text{etc.}$$

to a (n terms) $= s$;

therefore $(2a + \overline{n - 1}\,b) + (2a + \overline{n - 1}\,b) +$ etc.

(n terms) $= 2s$;

therefore $(2a + \overline{n - 1}\,b) \times n = 2s$

and $(2a + \overline{n - 1}\,b) \times \dfrac{n}{2} = s$.

(13) Of Geometrical Progressions

61. Quantities are said to be in geometrical progression, or continual proportion, when the first is to the second as the second to the third, and as the third to the fourth, etc.

Or when every succeeding term is a certain multiple or part of the preceding term.

Thus 8, 12, 18, 27 are in continued proportion or in geometric progression. In this case the terms are

$$8,\ 8 \times \frac{3}{2},\ 8 \times \frac{3}{2} \times \frac{3}{2},\ 8 \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2},$$

$$\text{or } 8,\ 8 \times \frac{3}{2},\ 8 \times \left(\frac{3}{2}\right)^2,\ 8 \times 8 \times \left(\frac{3}{2}\right)^3,$$

In like manner, a, ar, ar^2, ar^3 are in geometric progression.

62. The multiplier by which each term is obtained from the preceding is called the *common ratio*,

63. To find any term of a geometrical progression, multiply the first term by that power of the common difference which has for its exponent the number of the term *minus* one.

Thus the 5th term of the progression 8, 12, 18, etc. is,

$$8\left(\frac{3}{2}\right)^4 = 8 \times \frac{81}{16} = \frac{81}{2} = 40\tfrac{1}{2}\ .$$

And the n^{th} term of a, ar, ar^2, etc. is ar^{n-1}.

64. To find the sum of an increasing geometrical progression, multiply the last term by the common ratio, subtract from the product the first term, and divide the remainder by the excess of the common ratio above unity.

Thus, the sum of 5 terms of 8, 12, 18. etc. is

$$\frac{\frac{81}{2} \times \frac{3}{2} - 8}{\frac{3}{2} - 1} = \frac{243 - 32}{4} \div \frac{1}{2} = \frac{211}{2} = 105\tfrac{1}{2}.$$

And the sum of n terms of a, ar, ar^2, etc. is

$$\frac{ar^{n-1} \times r - a}{r - 1} = \frac{ar^2 - a}{r - 1}.$$

For if $a + ar + ar^2 + etc. + ar^{n-1}$ (n terms) $= s$,

multiplying by r, $ar + ar^2 +$ etc. $+ ar^{n-1} + ar^n$ (n terms) $= rs$,

and subtracting, $ar^n - a = rs - s = (r - 1)\ s$,

$$\text{whence } \frac{ar^n - a}{r - 1} = s.$$

Geometry.

Elements of Geometry. Euclid, Books *1, *2, *3, 4.

Book 5. Definition of Proportion.

The first of four magnitudes is said to have the same ratio to the second which the third has to the fourth when—any *equi*-multiples whatsoever of the *first* and *third* being taken, and any equi-multiples *whatsoever* of the *second* and *fourth*,—if the multiple of the first be *less* than that of the second, the multiple of the third is also less than that of the fourth; or if the multiple of the first be *equal* to the multiple of the second, the multiple of the third is also *equal* to that of the fourths or if the multiple of the first be *greater* than that of the second, the multiple of the third is also greater than that of the fourth.

Ratio is the relation of quantities in respect of proportion, so that if a, b, c, d be proportional, the ratio of a to b is equal to the ratio of c to d,

*Lemma 1. If magnitudes be proportionals according to the algebraical definition of proportion, they are also proportionals according to the geometrical definition.

If magnitudes a, b, c, d be proportionals algebraically, $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{ma}{nb} = \frac{mc}{nd}$, where ma, mc are any equi-multiples whatsoever of a, c, and nb, nd, any equi-multiples whatsoever of b, d; and if ma be less than nb, mc is less than nd; and if equal, equal; and if greater, greater. (Article 38.) Therefore, the magnitudes a, b, c, d are proportionals according to the geometrical definition.

Lemma 2. If magnitudes be proportionals according to the geometrical definition, they are also proportionals according to the algebraical definition.

If a : b :: c : d according to the geometrical definition, suppose, first, a to be any multiple, part, or parts of b so that $a = \frac{n}{m} b$; therefore $ma = nb$; therefore, by the definition $mc = nd$; therefore $\frac{c}{d} = \frac{n}{m}$; but $\frac{a}{b} = \frac{n}{m}$; therefore $\frac{a}{b} = \frac{c}{d}$.

Hence $\frac{a}{b} = \frac{c}{d}$, whenever a is any multiple, part, or parts of b. But when the quantities a, b, c, d are determined by any geometrical conditions, the fractions $\frac{a}{b}$ and $\frac{c}{d}$ will be equal or unequal according to those conditions, and the algebraical equation will express the results of these conditions generally, without regard to magnitude. Therefore, the equality cannot depend upon that particular magnitude of a or b, which makes a some multiple, part, or parts of b, Therefore, since, for those magnitudes of a and b for which a is a multiple, part, or parts of b, $\frac{a}{b}$ is equal

to $\frac{c}{d}$, these fractions must be equal without any such restriction, and we shall have in all cases $\frac{a}{b} = \frac{c}{d}$.

Hence when quantities have been proved to be geometrically proportional, we may apply to them all those results of algebraical proportion which have been already proved, in Articles 38 to 50.

EUCLID, Book 6.

Definition 1. The altitude of any figure is the straight line drawn from the vertex perpendicular to the base.

Definition 2. Similar rectilineal figures are those which have their several angles respectively equal, and the sides about the equal angles respectively proportionals.

*Proposition 1. Triangles and parallelograms of the same altitude are to one another as their bases.

*Proposition 2. If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or those produced, proportionally; and if the sides, or the sides produced, be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.

*Proposition 3. If the angle of a triangle be bisected by a straight line cutting the base, the segments of the base shall have the same ratio as the other sides of the triangle; and if the segments of the base are to each other as the other sides of the triangle, the straight line drawn from the vertex to the point of section, bisects the vertical angle.

Proposition A. If the exterior angle of a triangle, made by producing one of its sides, be bisected by a straight line, which also cuts the base produced; the segments between the dividing line and the extremities of the base are to each other as the other sides of the triangle; and if the segments of the base produced are to each other as the other sides of the triangle, the straight-line drawn from the vertex to the point of section divides the exterior angle of the triangle into two equal angles.

*Proposition 4. The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides; that is, are the antecedents or consequents of the ratios.

Corollary: to Proposition 4. Since it has been shown (Lemma 2) that when quantities are proportionals geometrically, they are proportionals algebraically; all the consequences which are proved of algebraical proportion (Articles 37 to 50) may be asserted of the proportionals in Propositions 1, 2, 3, A, 4 of this Book 6.

EUCLID. Book 11.

Definition 1. A straight line is perpendicular or at right angles to a plane, when it makes right angles with every straight line meeting it in that plane.

Definition 2. A plane is parallel to another plane when they do not meet, though both are indefinitely produced.

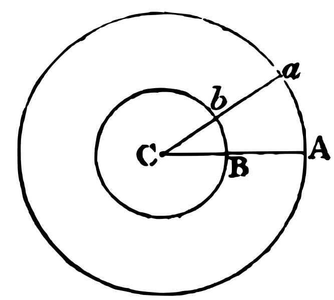
Definition 3. A plane is parallel to a straight line when they do not meet, though both are indefinitely produced.

Definition 4. A *prism* is a solid figure contained by two parallel planes, and by a number of other planes all parallel to one straight line, and cutting the first two planes so as to form polygons.

The first two planes are called the *ends* or *bases* of the prism, and the intermediate portion of the straight line to which all the other planes are parallel is the *length* of the prism.

The following Lemmas will be taken for granted: (straight lines, surfaces and solids being measured by numbers.)

Lemma 3. The arcs which subtend equal angles at the centers of two circles are as the radii of the circles.



Let the two circles be placed so that their centers coincide at *C*: and so that one of the lines *CA* containing the angle *ACa* in one of the circles coincides with the corresponding line *CB* in the other circle. Then the other lines containing the angles, namely *Ca*, *Cb*, will coincide; and it will be true that $Aa : Bb :: CA : CB$.

Lemma 4. The area of a rectangle is equal to the product of the two sides.

If *A*, *B* be the two sides, the rectangle is $= A \times B$.

Corollary: If *B* be the base and *A* the altitude of a triangle, the area of the triangle is $= \frac{1}{2} A \times B$.

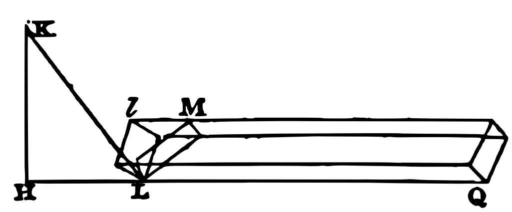
Lemma 5. If a prism be cut by planes perpendicular to its length at different points, the areas of the sections are all similar and equal.

Lemma 6. The solid content of a prism is equal to the product of its length and of the area of a section perpendicular to the length.

If *A* be the area of the section and *H* the length, the solid content is $= A \times H$. In this case, solid contents are measured by the number of times they contain a unit of solid content.

Corollary: In a uniform prism the weight is as the solid content; hence the weight of any portion of a uniform prism is proportional to its length.

Lemma 7. If a prism be cut by two planes passing through any point of its length, one of the planes being perpendicular to the length and the other oblique to it; and if a line be drawn at the point, perpendicular to the oblique section and intercepted by a line perpendicular to the



length; the oblique section is to the perpendicular section as the portion of the perpendicular line intercepted is to the portion of the length intercepted.

Let *Ll*, *LM* be the perpendicular and the oblique section of the prism, of which the length is *QL*, *LK* perpendicular to the section *LM*, and *KH* perpendicular to the length QL. Then area *LM* : area *Ll* :: *KL* : *HL*.

Mechanics.

Note. JC. As Plato noted, we are born with what in later history has been called a priori knowledge, i.e., the ability of sense to understand equal, greater, and less. These are independent of any particular system of grammar and any kind of material difference and this recognition should be exercised. We exercise it by asking and answering questions. These words should be exercised in the demonstrations instead of such silly rubbish words like preponderate. With the leaver, you can show perceptibly the step-by-step construction of the whole of rational thought, which is independent of any particular system of Grammar. These are not laws of this or that, these laws follow from the recursion of equal, greater, and less, universally. What we have to learn now, is how to teach it in a proper and clear order.

You can not only move the world with a leaver, you can move the mind of man to learn and do his own work. Simply pair the perceptible with the intelligible.

There are no laws of this or that, there is only methods of information processing via systems of grammar. This system of grammar becomes our Grammar Matrix of Common Grammar, Arithmetic, Algebra and Geometry. See if you can find in this work, any other. The intelligible binary multiplied by the perceptible binary, can only produce four specific systems of Grammar.

The a priori knowledge we are born with, the ability to comprehend equal, greater and less, should not be taught in college, but in kindergarten. The toys a child should have, are those toys with make them ask and answer questions, guided by an intelligent parent.

Posteriori knowledge is acquired from our environment, we are born with the intelligible, the noun, and are given by our environment, the verb, the material differences to apply that knowledge to. It teaches us how to add and subtract, multiply and divide, all starting from the recursion of equal, greater and less.

Book I. Statics.

Definitions and fundamental notions.

1. Mechanics is the science which treats of the laws of the motion and rest of bodies.

2. Any cause which moves or tends to move a body, or which changes or tends to change its motion, is called *force*.

3. *Body* or *Matter* is anything extended, and possessing the power of resisting the action of force.

A *rigid* body is one in which the force applied at one part of the body is transferred to another part, the relative positions of the parts of the body not being capable of any change.

4. All bodies within our observation fall or tend to fall to the earth : and the force which they exert in consequence of this tendency, is called their weight.

5. Forces may produce cither rest or motion in bodies. When forces produce rest, they *balance* each other; they are in *equilibrium*; they *cancel out* each other's effects.

6. *Statics* is the science which treats of the laws of forces in equilibrium.

7. Two directly opposite forces which balance each other are *equal*.

Forces are directly opposite when they act in the same straight line in opposite directions.

8. Forces are capable of *addition*. Thus, when two men pull at a string in the same direction, their forces are added; and when two heavy bodies are put in the

same vessel suspended by a string, their weights are added, and are supported by the string.

9. A force is *twice* as great as a given force, when it is the sum of two others, each equal to the given force; a force is *three* times as great, when it is the sum of three such forces; and so on.

10. Forces (in Statics) may be measured by the weights which they would support.

11. The *Quantities of Matter* of bodies are measured by the proportion of their mechanical effect.

12. The quantities of matter of two bodies are *as their weight* at the same place.

13. The *Density* of a body is measured by the quantity of matter contained in a given space.

Section 1. The Lever. Definitions.

1. A *Lever* is a rigid rod, moveable, in one plane, about a point, which is called the *fulcrum* or *center of motion*, by means of forces which tend to turn it round the fulcrum.

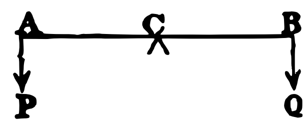
2. The portions of the rod between the fulcrum and the points where the forces are applied, are called the *arms*.

3. When the arms are two portions of the same straight line, the lever is called a *straight* lever; otherwise, it is called a *bent* lever.

4. The lever is supposed to be without weight, unless the contrary be expressed.

AXIOMS.

1. If two equal forces act perpendicularly at the extremities of equal arms of a straight lever to turn it opposite ways, they will keep each other in equilibrium.

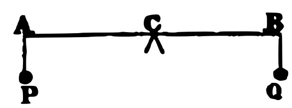


If $AC = BC$, and P and Q be two equal forces acting perpendicularly on CA and CB at A and B, they will balance each other.

2. If forces keep each other in equilibrium, and if any force be added to one of them, it will be greater.

Corollary: Hence the converse of Axiom 1 is true; if two forces P, Q acting perpendicularly at equal arms balance, they are equal. For if they are unequal, let $P + X = Q$; then $P + X$ will balance Q, by Axiom 1; but since P balances Q, $P + X$ will be greater by Axiom 2 : which is absurd. Therefore $P = Q$.

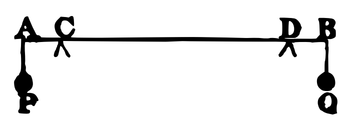
3. If two equal weights balance each other upon a horizontal straight lever, the pressure upon the fulcrum is equal to the sum of the weights, whatever be the length of the lever.

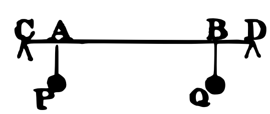


If P, Q be two equal weights which balance each other upon the horizontal lever AB, the pressure upon C is $P + Q$.

Corollary: If two equal forces acting perpendicularly on the arm of a straight lever balance, the pressure on the fulcrum is equal to the sum of the forces. For (Definition 10) all statical forces are equal to the weights which they would support; and hence, if for the weights, be substituted the forces which would support them, the pressure on the fulcrum is not altered.

4. If two equal weights be supported upon a straight lever on two fulcrums at equal distances from the weights, the pressures upon the two fulcrums are together equal to the sum of the weights.

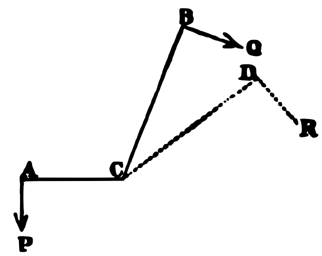




If P, Q be two equal weights which are supported upon the line AB on two fulcrums C, D, so that AC, BD are equal; the pressures upon C, D are together equal to the sum of the weights $P + Q$.

5. On the same suppositions, the pressures on the two fulcrums are equal.

6. If a force act perpendicularly on the straight arm of a bent lever at its extremity, the effect to turn the lever round the fulcrum will be the same, whatever be the angle which the arm makes with the other arm, so long as the length is the same.



If a force Q act perpendicularly on CB at its extremity B, C being the fulcrum, and an equal force R act perpendicularly on an equal arm CD, at its extremity, the effect to turn the lever round C in the two cases is equal.

7. When a force acts upon a rigid body it will produce the same effect to urge the body in the line of its own direction, at whatever point of the direction it acts.

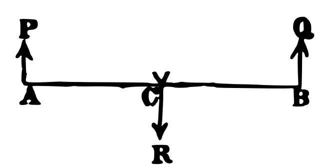
8. If a body which is moveable about an axis be acted upon by two equal forces, in two planes perpendicular to the axis, the forces being perpendicular at the extremities of two straight arms of equal length from the axis; the two forces will produce equal effects to turn the body, at whatever points the arms meet the axis.

9. If a stretched string pass freely round a fixed body, so that the direction of the string is altered, any force exerted at one extremity of the string will produce at the other extremity the same effect as if the force had acted directly.

10. If .in a system which is in equilibrium, there be substituted for the force acting at any point, an immoveable fulcrum at that point, the equilibrium will not be disturbed.

11. If in a system which is in equilibrium there be substituted for an immoveable point or fulcrum the force which the fulcrum exerts, the equilibrium will not be disturbed.

Corollary: If a weight be supported on a horizontal rod by two forces acting vertically at equal distances from the weight, the forces are equal to each other, and their sum is equal to the weight. For, let two forces P, Q balance each other, acting perpendicularly on the equal arms of a lever AB: then by Corollary: to Axiom 2, they are equal. Also, by Corollary: to Axiom 3, the pressure upon the fulcrum is equal to the sum of the forces P, Q. Hence, by Axiom 11, if instead of a fulcrum, there be a force R, acting at C perpendicular to the lever, and equal to the sum of P and Q, this force will balance the pressure at C, just as the fulcrum does, and there will be an equilibrium; that is, a vertical force, or weight R, will be supported by two forces equal to P, Q acting vertically at equal distances CA, CB; and the weight R is equal to $P + Q$.



12. A perfectly hard and smooth surface, acted on at any point by any force, exerts a reaction which is perpendicular to the surface at that point; and if the surface be supposed to be immoveable, the force will be supported, whatever be its magnitude.

13. A heavy material straight line, prism, or cylinder, of uniform density, may be supposed to be composed of a row of heavy points of equal weight, uniformly distributed along the line.

14. A heavy material plane of uniform density may be supposed to be composed of a collection of parallel straight lines of equal density, uniformly distributed along the plane.

15. A heavy solid body of uniform density may be supposed to be composed of a collection of particles, the weight of each of which is as the portion of the body which it occupies; and which particles may be considered as heavy points.

POSTULATES.

1. A prism or cylinder of uniform density, and of given length, may be taken, which is equal to any given weight.

2. A force may be taken equal to the excess of a greater given force over a less.

3. A force may be taken in a given ratio to a given force.

REMARKS ON THE AXIOMS OF STATICS.

1 The Axioms of Statics in the preceding pages are simply-stated, without addition or explanation; in the same manner in which the Axioms of Geometry are stated in Treatises on Geometry. As the Axioms of Geometry are derived from the idea of space, so the Axioms of Statics are derived from the idea of *statical force* or *pressure*, and the idea of *body* or *matter,* as that which receives and transmits pressure. The student must possess distinctly this idea of force acting upon body, and body sustaining force of body resisting the action of force, and while it resists, transmitting this action; —of body with this mechanical property, existing in the various forms of rigid straight line, lever, plane, solid, flexible line, flexible surface; —and when he has this distinct possession of these elementary ideas, the truth of the Axioms of Statics will be seen as self-evident, and he will be in a condition to go on with the reasonings by which the following Propositions are established.

But we may make a few Remarks tending to illustrate the self-evident character of the above Axioms.

2. We shall begin with the consideration of the First Axiom of Statics; which is, "If two equal forces act perpendicularly at the extremities of equal arms of a straight line to turn it opposite ways, they will keep each other in equilibrium." This is often, and properly, further confirmed, by observing that there is no reason why one of the forces should be greater rather than the other, and that, as both cannot be greater, neither will do so. All the circumstances on which the result (equilibrium or greater and less) can depend, are equal on the two sides;—equal arms, equal angles, equal forces. If the forces are not in equilibrium, which will be greater and which will be less? No answer can be given, because there is no circumstance left by which either can be distinguished.

3. The argument which we have just used, is often applicable, and may be expressed by the formula, "there is no reason why one of the two opposite cases should occur, which is not equally valid for the other; and as both cannot occur (for they are opposite cases) neither will occur." This argument is called "the principle of sufficient reason it puts in a general form the considerations on which

several of our axioms depend; and to persons who are accustomed to such generality, it may make their truth clearer.

The same principle might be applied to other cases, for example, to Axiom 6, that the effect produced on a bent lever does not depend on the direction of the arm. For if we suppose two forces acting perpendicularly on two equal arms of a bent lever to turn it opposite ways, these forces will balance, whatever be the angle which they make, since there is no reason why either should be greater: it thus appears, that the force which, acting at *A*, would be balanced by *Q* in the figure to Axiom 6, would also be balanced by *R*, and therefore these two forces produce the same effect; which is what the axiom asserts.

4. The same reasoning might be applied to Axiom 8; for if two equal forces act at right angles at equal arms, in planes perpendicular to the axis of a rigid body, and tend to turn it opposite ways, they will balance each other, since all the conditions are the same for both forces.

5. Nearly the same may be said of Axiom 9;—if a stretched string pass freely round a fixed body, equal forces acting at its two ends will balance each other; for if it passes with perfect freedom, its passing round the point cannot give an advantage to either force. Therefore, the force which will be balanced by the string at its second extremity is exactly equal to the force which acts at its first extremity. The same principle may be applied to prove Axiom 5.

6 The axioms which are perhaps least obvious are Axioms 3 and 4; for instance, the former;—that "the pressure upon the fulcrum is equal to the sum of the weights." Yet this becomes evident when we consider it steadily. It will then be seen that we conceive pressure or weight as something which must be supported so that the whole support must be equal to the whole pressure. The two weights which act upon the lever must be somehow balanced and counteracted, and the length of the lever cannot at all remove or alter this necessity. Their pressure will be the same as if the two arms of the lever were shortened till the weights coincided at the fulcrum; but in this case, it is clear that the pressure on the fulcrum would be equal to the sum of the weights: therefore, it will be so in every other case.

7. This principle, that in cases of statical equilibrium, a force is necessarily supported by an equal force, is sometimes expressed as an Axiom, by saying that "Action is always accompanied by an equal and opposite Re-action." This principle thus stated may be considered as an expression of the conception of equality as applied to forces; or as a Definition of *equal forces*. This principle is implied in the conception of any comparison of forces; for equilibrium and addition of forces are modes in which forces are compared, as superposition and addition of distances are modes in which geometrical quantities are compared.

We may further observe, that this fundamental conception of action and re-action is equivalent to the conception of force and matter, which are ideas necessarily connected and correlative. Matter, as stated in page 26, is that which can resist the action of force. In Mechanics at least, we know matter only as the subject on which force acts.

8. But matter not only receives, it also transmits the action of force; and it is impossible to reason respecting the mechanical results of such transmission, without laying down the fundamental principle by which it operates. And this accordingly is the purpose of Axioms 6, 7, 8, 9, 10, 12. When the body is supposed to be perfectly rigid, it transmits force without any change or yielding. This rigidity of a body is contemplated under different aspects, in the Axioms just referred to. In Axiom 7, it is the rigidity of a rod pushed endways; in Axiom 8, the rigidity of a plane turned about a fixed-point; in Axiom 8, the rigidity of a solid twisted about an axis. Axiom 9 defines the manner in which a flexible string transmits pressure, and in like manner we shall have Axioms in Hydrostatics, defining the manner in which a fluid transmits pressure. We may call Axioms 6, 7, 8, collectively, the Definition of a *rigid body*. The place of these principles in our reasoning will not be thereby altered; nor will the necessity of their being accompanied by distinct mechanical conceptions be superseded.

9. Axioms 13, 14, 15, of the Statics, are all included in the general consideration, that material bodies may be supposed to consist of material parts, and that the weight of the whole is equal to the weight of all the parts; but they are stated separately, because they are used separately, and because they are at least as evident in these more particular cases as they are in the more general form.

By considerations of this nature it appears, that the axioms, as above stated, are evident in their nature, in virtue of the conceptions which we necessarily form, in order to reason upon mechanical subjects.

10. Some persons may be surprised to find the Axioms of Mechanics represented as so numerous; Especially if they look for analogy to Geometry, where the necessary axioms are confessedly few, and according to some writers, none; and they may be led to think that many of the axioms here given must be superfluous, by observing that in most mechanical works the fundamental principles are stated as much fewer than these. But very few of those which are here stated are superfluous in effect. From the very circumstance that they are axioms, they are assented to when they are adduced in the reasoning, whether they have been before asserted or not; but to make our reasoning formally correct every proposition which is assumed should be previously stated. And when we consider carefully, we see that the various modifications and combinations of the ideas of force, body, and equilibrium, along with the idea of space of one, two, or three dimensions, readily branch out into as many heads as appear in this part of the present work.

11. Some persons may be disposed at first to say, that our knowledge of such elementary truths as pre stated in the Axioms of Statics and Hydrostatics, is collected from observation and experience. But in refutation of this we may remark, that we cannot experimentally verify these elementary truths, without assuming other principles, which require proof as much as these do. If, for instance, Archimedes had wished to ascertain by trial whether two equal weights at the equal arms of a lever would balance each other, how could he know that the weights were equal, by anymore simple criterion than that they *did* balance? But in fact, it is perfectly certain that of the thousands of persons who from the time of

Archimedes to the present day have studied Statics as a mathematical science, a very few have received or required any confirmation of his axioms from experiment; and those who have needed such help have undoubtedly been those in whom the apprehension of the real nature and force of the evidence of the subject was most obscure.

12. We do not assert that the axioms as stated in this Treatise are given in the only exact form; or that they may not be improved, simplified, and reduced in number. But it does not seem likely that this can be done to any great extent, consistently with the rigor of deductive proof. The Fourth Axiom of Statics is one which attempts have been made to supersede: for example, Lagrange* has endeavored to deduce it from the preceding ones. But it will be found that his proof, if distinctly stated, involves some such axiom as this:—that "If two forces, acting at the extremities of a straight line, and a single force, acting at an intermediate point of the straight line, produce the same effect to turn a body about another line, the two forces produce at the intermediate point an effect equal to the single force." And though this axiom may be self-evident, it will hardly be considered as simpler than that which it replaces.

13. Thus, the science of Statics, like Geometry, rests upon axioms which are neither derived directly from experience, nor capable of being superseded by definitions, nor by simpler principles. In this science, as in Geometry, the evidence of these fundamental truths resides in those convictions to which an attentive and steady consideration of the subject necessarily leads us. The axioms with regard to pressures, action, and re-action, equilibrium and preponderances, rigid and flexible bodies, result necessarily from the conceptions which are involved in all exact reasoning on such matters. The axioms do not flow from the definitions, but they flow irresistibly along with the definitions, from the distinctness of our ideas upon the subjects thus brought into view. These axioms are not arbitrary assumptions, nor selected hypotheses; but truths which we must see to be necessarily and universally true, before we can reason on to anything else; and in Mechanics, as in Geometry, the capacity of seeing that they are thus true, is required in the student, in order that he and the writer may be able to proceed together.

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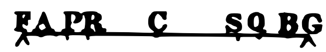
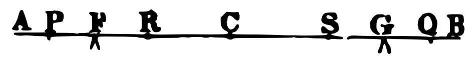
* Mecanique Analytique. Introduction.

Propositions.

N B. The Propositions required by the University for the degree of B. A are those which are marked by numbers; and the Enunciations are printed in larger type.

Proposition 1. A horizontal prism or cylinder of uniform density will produce the same effect by its weight as if it were collected at its middle point.





Let *AB* be the prism or cylinder, and C its middle point. Let P, R be any points in *AC*, and let *CQ* be taken equal to *CP*, and *CS* equal to *CR*.

The half *AC* of the prism may (by Axiom 13.) be supposed to be made up of small equal weights, distributed along the whole of the line *AC*, as at *P*, *R*; and the half *BC* may in like manner be conceived to be made up of small equal weights distributed along *BC*; as at *Q*, *S*; of which the weight at *Q* is equal to the weight at *P*, that at *S* to that at *R*, and so on.

Let *F* be a fulcrum about which the prism *AB* tends to turn by its weight. In *CB*, produced, if necessary, take *CG* equal to *CF*, and suppose a fulcrum placed at *G*.

Let the weights at *P*, *Q*, *R*, *S* be denoted by *P*, *Q*, *R*, *S*.

The two weights *P* and *Q* produce upon the fulcrums *F* and *G* pressures which together are equal to the sum of the weights $P + Q$, (Axiom 4,) or to the double of *P*, since *P* and *Q* are equal. But the pressure upon each of these fulcrums is equal, (Axiom 5,) hence the pressure upon each of them is *P*; therefore, the pressure upon the fulcrum *G*, arising from the two weights *P* and *Q*, is *P*; in like manner the pressure upon the fulcrum *G*, arising from *R* and *S*, is *R*; and so of the rest: and the whole pressure on *G*, arising from the whole prism *AB*, is the sum of all the weights *P*, *R*, etc. from *A* to *C*; that is, it is half the weight of the prism.

But if the whole prism be collected in its middle point *C*, the pressure upon the two fulcrums *F* and *G* will be the whole weight of the prism, and the pressures on the two fulcrums are equal; by Corollary: to Axiom 11. Therefore, in this case also, the pressure on the fulcrum *G* is equal to half the weight of the prism. Therefore, the prism, when collected at its middle point, produces the same pressure on the fulcrum *G* as it did before.

Therefore, when a uniform prism is collected at its middle point, it produces the same effect by its weight as it did before, Q. E. D.

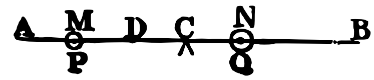
Corollary: 1. A uniform prism or cylinder will balance itself upon its middle point.

Corollary: 2. When a prism or cylinder thus balances upon its middle point, the pressure upon the fulcrum is equal to the weight of the prism.

Proposition 2. If two weights acting perpendicularly at the extremities of the arms of a [horizontal] straight lever on opposite sides of the fulcrum balance each other, they are inversely as their distances from the fulcrum, and the pressure on the fulcrum is equal to their sum.

Let P, Q be the two weights.

Let there be a uniform prism of the length AB, equal in weight to $P + Q$ (Postulate 1), and let $AD : DB :: P : Q$. Therefore, componendo, $AD + BD : AD :: P + Q : P$. But AD + BD is equal in weight to P + Q, and the prism is uniform; therefore, by Corollary: to Lemma 6, the prism AD is equal in weight to P. In like manner the prism BD is equal in weight to Q.



Let C be the middle point of AB; M, the middle point of AD; N, the middle point of DB. By Proposition 1. Corollary: 1 and 2, the prism AB will balance on the point C, and the pressure on that point will be equal to the weight of the prism, that is to $P + Q$.

But by Proposition 1. the prism AD will produce the same effect as if it be collected at its middle point M; that is, the same effect as the weight P at M. And in like manner the prism DB will produce the same effect as the weight Q at N. Therefore, the whole prism AB will produce the same effect as the weight P at M, and the weight Q at N; that is, the weight P at M, and Q at N will balance on C.

But since MD is half AD, and DN is half DB, the sum MN is half the sum AB, and is therefore equal to AC. Hence taking away the common part MC, the remainder CN is equal to AM, or MD. And to MD and CN adding the common part DC, MC is equal to DN.

Now $P : Q :: AD : DB$ by construction; that is,

$P : Q :: 2MD : 2DN$; or $:: MD : DN$;

hence, by what has been proved,

$$P : Q :: CN : MC.$$

Therefore, the weights P, Q are inversely as their distances from the point C on which they balance. Q. E. D.

Also, the weights P, Q collected at M, N produce the same effect on the fulcrum C as the prisms AD, DB; that is, as the prism AB; that is, they produce a pressure $P + Q$, as has been shown, Q. E. D.

Corollary: If two *forces* acting perpendicularly on a straight lever on opposite sides of the fulcrum balance each other, they are inversely as their distances from the fulcrum, and the pressure on the fulcrum is equal to the sum of the forces.

For any forces may be represented by weights; and what is true of the weights is true of the forces.

Proposition A. If two weights acting perpendicularly at the extremities of the arms of a straight horizontal lever on opposite sides of the fulcrum are inversely as their distances from the fulcrum, they will balance each other.

As in the last Proposition, let there be a uniform prism AB, equal in weight to the sum of the weights $P + Q$; and let it be divided in D, so that $AD : DB :: P : Q$; then, as before, AD is equal in weight to P, and BD to Q.

Let M be the middle point of AD; N, of DB. And let C be a point, such that $CN : MC :: P : Q$.

Then $CN : MC :: AD : DB$

$$:: 2MP : 2DN :: MD : DN,$$

whence $MC + CN : CN :: MD + DN : MD$,

and the first and third are equal; therefore, CN is equal to MD.

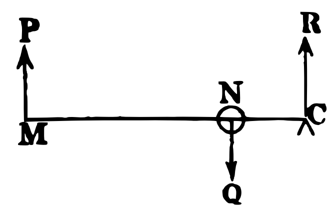
Hence adding DC to both, MC is equal to DN or NB; and hence AM and MC together are equal to CN and NB together; that is, AC is equal to CB; and C is the middle point of AB.

Therefore, the prism AB will balance on C; and by Proposition 1. if the part AD, that is, P, be collected at M, and the part DB, that is, Q, be collected at N, the effect will still be the same; that is, P and Q will balance on C. Therefore, etc. Q. E. D.

Corollary: 1. In this case also the pressure upon the fulcrum C is equal to $P + Q$,

Corollary: 2. If for weights be put any forces, the lever being in any position, the same proposition is true.

Proposition 3. If two forces acting perpendicularly on a straight lever in opposite directions and on the same side of the fulcrum balance each other, they are inversely as their distances from the fulcrum; and the pressure on the fulcrum is equal to the difference of the forces.



Let MCN be the lever on which the two forces P and Q acting perpendicularly at M and N in opposite directions balance each other. Let R be a force such that $P + R$ is equal to Q, and let MNC be supposed to be a lever on which two forces P, R, acting perpendicularly at M, C on opposite sides of the fulcrum, balance each other. Then, by Proposition 2. the pressure upon the fulcrum N is equal to $P + R$, that is to Q, and is in the direction of the forces P and R. Hence if a force $P + R$, that is Q, act perpendicularly to the lever MC at N in the direction opposite to P and R, it will supply the place of the fulcrum, and the forces, P, Q, R will still balance each other by Axiom 11. And if we place an immoveable fulcrum ‘at C, it will supply the place of the force R, and the forces P, Q, will still balance each other by Axiom 10.

But since P, R balance on the lever MNC, we have by Proposition 2.

$$R : P :: MN : NC; \text{ and therefore}$$

$$R + P : P :: MN + NC : NC; \text{ that is}$$

$$Q : P :: MC : NC;$$

the forces P, Q are inversely as their distances from the fulcrum C.

Also, the pressure on the fulcrum C, which replaces the force R is equal to the force R, that is to the difference of the forces P and Q. Q. E. D.

Proposition 4. To explain the different kinds of levers.

When material levers are used, the two forces which have been spoken of, as balancing each other upon the lever, are exemplified by the weight to be raised or the resistance to be overcome, as the one force, and the pressure, weight, or force

of any kind, employed for the purpose, as the other force. The former of these forces is called *the Weight*. the latter is called *the Power*.

The preceding Propositions give the proportion of the Power and Weight in the case of equilibrium, that is, when the weight is not raised, but only supported; or when the resistance is not overcome, but only neutralized. But knowing the Power which will produce equilibrium with the weight, we know that any additional force will make the Power greater. (Axiom 2.)

Straight levers are divided into three kinds, according to the position of the Power and Weight.

1. The Lever of the First kind is that in which the Power and Weight are on opposite sides of the Fulcrum, as in Proposition 2. and A.

We have an example of a lever of this kind, when a bar is used to raise a heavy stone by pressing down one end of the bar with the hand, so as to raise the stone with the other end: the Power is the force of the hand, the Fulcrum is the obstacle on which the bar rests, the Weight is the weight of the stone.

We have an example of a double lever of this kind in a pair of pincers used for holding or cutting; the Power is the force of the hand or hands at the handle, the Weight is the resistance overcome by the pinching edges of the instrument, the Fulcrum is the pin on which the two pieces of the instrument move.

2. The Lever of the Second kind is that in which the Power and the Weight are on the same side of the Fulcrum, the Weight being the nearer to the Fulcrum.

We have an example of a lever of this kind, when a bar is used to raise a heavy stone by raising one end of the bar with the hand, while the other end rests on the ground, and the stone is raised by an intermediate part of the bar. The Fulcrum is the ground, the Power is the force exerted by the hand, the Weight is the weight of the stone.

We have an example of a double lever of this kind in a pair of nutcrackers. The Power is the force of the hand exerted at the handles; the Weight is the (force with which the nut resists crushing; the Fulcrum is the pin which connects the two pieces of the instrument.

3. The Lever of the Third kind is that in which the Power and the Weight are on the same side of the fulcrum, and the Weight is the further from the fulcrum.

In this kind of lever, the Power must be greater than the weight in order to produce equilibrium, by Proposition 3. Therefore, by the use of such a lever, force is lost. The advantage gained by the lever is, that the force exerted produces its effect Ut an increased distance from the fulcrum.

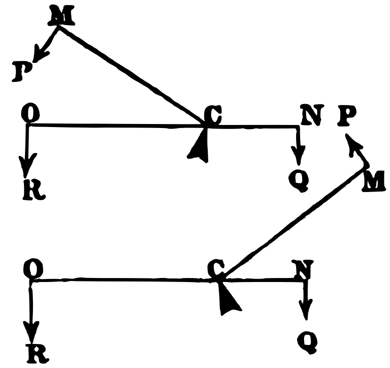
We have an example of a lever of this kind in the anatomy of the fore-arm of a man, when he raises a load with it, turning at the elbow. The elbow is the Fulcrum, the Power is the force of the muscle which, coming from the upper arm is inserted into the forearm near the elbow, the Weight is the load raised.

We have an example of a double lever of this kind in a pair of tongs used to hold a coal. The Fulcrum is the pin on which the two parts of the instrument turn,

the Power is the force of the fingers, the Weight is the pressure exerted by the coal upon the ends of the tongs.

Proposition 5. If two forces acting perpendicularly at the extremities of the arms of any lever balance each other, they are inversely as the arms.

Let *MCN* be any lever: and let *P*, *Q* acting perpendicularly on the arms *CM*, *CN* balance each other; then $P : Q :: CN : CM$.

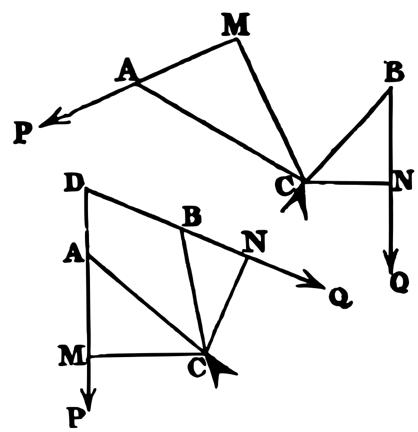


Produce *NC* to *Q*, taking *CO* equal to *CM*; and at *O* let a force *R* equal to *P* act perpendicularly on the lever *NCO*, to turn it in the same direction as *P*. Then since *CM* is equal to *CO*, and therefore *P* to the force *R*, both acting perpendicularly to the arms, by Axiom 6, *P* and *R* will produce the same effect to turn the lever round the fulcrum *C*; and therefore, since *P* balances *Q*, *R* will balance *Q*.

But since forces *R* and *Q* balance on the straight lever *OCN*, by Proposition 2. $R : Q :: CN : CO$; and since *P* is equal to *R*, and *CO* to *CM*, $P : Q :: CN : CM$; or the forces *P*, *Q* are inversely as their arms. Q. E. D.

Proposition 6. If two forces acting at any angles on the arms of any lever balance each other, they are inversely as the perpendiculars drawn from the fulcrum to the directions in which the forces act.

Let *ACB* be the lever on which the forces *P*, *Q* acting at any angles balance each other; and let *CM*, *CN* be the perpendiculars from the fulcrum *C* in the directions of the forces; then $P : Q :: CN : CM$.



The lever *ACB* is supposed to be rigid, so that the arms *AC*, *BC* cannot alter their respective positions. Hence, we may suppose the plane ACB to be a right indefinite plane, moveable about the point *C*, and *AC*, *BC* to be lines in this plane. Therefore, the forces *P*, *Q*, which act at the points *A*, *B*, will by Axiom 7, produce the same effect as if they act at the points *M*, *N* respectively: therefore, if they act at these points *M*, *N* they will still balance.

Hence, by Proposition 5. $P : Q :: CN : CM$; or the forces are inversely as the perpendiculars *CM*, *CN*. Q. E. D.

Corollary: 1. The converse is true, that if $P : Q :: CN : CM$, the forces will balance.

Corollary: 2. If *P*, *Q*, *CM*, *CN* be expressed in numbers when *P*, *Q* balance, $P \times CM = Q \times CN$ and when $P \times CM = Q \times CN$, *P* and *Q* balance.

Definition of the moment of a force. If lines be expressed in numbers, the product which arises when a force acting on a lever is multiplied by the perpendicular from the fulcrum of the lever upon the direction of the force is called the *moment* of the force.

It appears by the last Corollary:, that when two forces balance on a lever, their moments are equal; and when their moments are equal, they balance.

Also, if the moment of one force be the greater, that force will be greater.

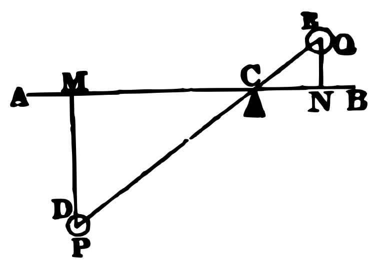
Corollary: 3. If X be any force acting on the lever ACB, and CO the perpendicular upon its direction, and if $X \times CO = P \times CM$, the force X will produce upon the lever the same effect as P. For $X \times CO = Q \times CN$; therefore, by this Proposition, X will balance Q; which is what P does.

Corollary: 4. If the two forces P, Q act at the same point D, the proposition is still true.

Proposition 7. If two weights balance each other on a straight lever when it is horizontal, they will balance each other in every position of the lever.

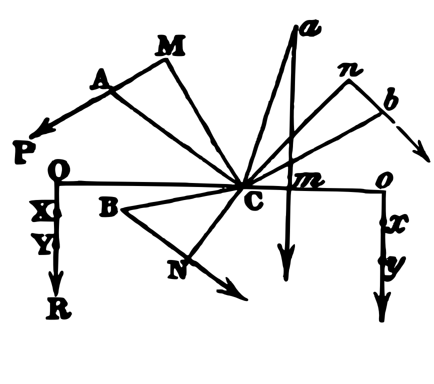
Let it be supposed that the weights P, Q, acting at A, B, balance each other upon the lever when it is in the horizontal position ACB; the weights P, Q will balance each other upon the same lever in any other position, as DCE.

Draw DM, EN vertical, meeting the horizontal line ACB. Then, in the triangles DCM, ECN, the vertical angles DCM, ECN are equal; and DMC, ENC are equal, being right angles; therefore, the remaining angles of the triangles are equal, and the triangles are equiangular and similar. Therefore, $DC : CM :: EC : CN$, and alternately $DC : EC :: CM : CN$. But since P, Q balance each other on AB, $Q : P :: AC : CB$; and AC is equal to DC, and CB to EC, because ACB an DCE are the same lever; therefore $Q : P :: DC : EC$; therefore, by what precedes, $Q : P :: CM : CN$; therefore, by Proposition 6. the weights P, Q, acting at the points D, E, will balance each other, Q. E. D.



Corollary: The pressure upon the fulcrum C in every position of the lever DE is equal to the sum of the weights P and Q. For in every position the effect of the weights P, Q is the same as if they acted at M, N, by Axiom 7. But in this case, by Proposition 2. the pressure on the fulcrum C is the sum of the weights.

Proposition B. If any number of forces act upon a lever, and tend to turn it opposite ways, and if the sum of the moments of the forces which tend to turn the lever one way be equal to the sum of the moments of the forces which tend to turn it the other way, the forces will balance each other.



Let the forces P, Q, R, tend to turn the lever one way, and let CM, CN, CO be the perpendiculars on their directions; and let the p, q, tend to turn the lever the other way, and let Cm, Cn be the perpendiculars on their directions; and let $P \times CM + Q \times CN + R \times CO$ be equal to $p \times Cm + q \times Cn$; the forces will, balance each other.

Let any two lines CO, Co be taken, and let forces act at O and o, perpendicularly to CO, Co, to turn the lever opposite ways, namely, at O, a force X, such that CO :

$CM :: P : X$, by Postulate 3. that is, such that $X \times CO = P \times CM$; and also a force Y, such that $Y \times CO = Q \times CN$, and a force R; and also at o, a force x, such that $x \times Co$, and a force y, such that $y \times Co = q \times Cn$.

Then, by Corollary: 3 to Proposition 6, the force X will produce the same effect as the force P, and the force Y will produce the same effect as the force Q; and therefore, the forces P, Q, R will produce the same effect as X, Y, R acting at O. In like manner the forces p, q will produce the same effect as x, y, acting at o.

But the forces X, Y, R, acting at O, will balance the forces x, y, acting at o, if $(X + Y + R) \times CO$ be equal to $(x + y) \times Co$, by Proposition 6; that is, if $X \times CO + Y \times CO + R \times CO$ be equal to $x \times Co + y \times Co$; that is, by the construction, if $P \times CM + Q \times CN + R \times CO$ be equal to $p \times Cm + q \times Cn$. Therefore, etc. Q. E. D.

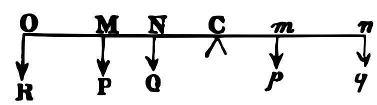
Corollary: 1. If the forces be weights acting on a straight horizontal lever, the same is true, putting for the perpendiculars on the directions of the forces, the portions of the lever CM, CN, etc. intercepted between the fulcrum and the weights. (See next figure).

Corollary: 2. The converse of this Proposition and of Corollary: 1 are true.

Proposition C. If any forces act perpendicularly upon a lever, the pressure on the fulcrum is equal to the sum of the forces.

It will first be proved that if any number of forces acting perpendicularly upon a lever balance each other, they may be separated into parts, so that, retaining their positions, they form pairs, each of which pairs would balance on the fulcrum separately.

Let P, Q, R, p, q be any forces which balance each other on the lever $O\ M\ N\ C\ m\ n$. If each force on one side of the fulcrum has its moment equal to that of a corresponding force on the other side, it is clear that each force will balance the corresponding one on the other side, and the forces are already in such pairs as are mentioned above. But if not, let any moment on one side, as $P \times CM$, be less than a moment on the other side, as $p \times Cm$. Assume a force u such that $Cm : CM :: P : u$, by Postulate 3: therefore, $P \times CM = u \times Cm$; therefore $u \times Cm$ is less than $p \times Cm$, and u is less than p; let $p = u + x$. Then if p be separated into parts u and x, the pair P and u will balance each other separately, because their moments are equal.



In the same manner, of the forces Q, R, x, q, take any other as Q, of which the moment $Q \times CN$ is less than the moment of $q \times Cn$ of a force q on the other side of the fulcrum. Assume a force v such, that $Cn : CN :: Q : v$, therefore $Q \times CN = v \times Cn$; and. let $q = v + y$. Then if q be separated into v and y, the pair Q and v will balance each other separately, for the same reason as before.

And of the forces R, x, y, the moment $x \times Cm$ must be less than $R \times CO$. Assume $X \times CO = x \times Cm$; and let $R = X + Y$. The pair X, x will balance each other separately, as before.

But because the forces P, Q, R, p, q balance on the lever, it follows (by Corollary: 2 to Proposition B) that

$$P \times CM + Q \times CN + R \times CO = p \times Cm + q \times Cn;$$

and hence since

$$R = X + Y, \text{ and } p = u + x, \text{ and } q = v + y,$$
$$P \times CM + Q \times CN + X \times CO + Y \times GO$$
$$= u \times Cm + x \times Cm + v \times Cn + y \times Cn;$$

and it has been supposed that

$$P \times CM = u \times Cm, \text{ and } Q \times CN = v \times Cn,$$
$$\text{and } X \times CO = x \times Cm;$$

hence the remainder

$$Y \times CO \text{ is } = y \times Cn;$$

and the pair Y, y will balance each other.

Therefore, the forces have been separated into pairs,

$$P, u; Q, v; X, x; Y, y;$$

which balance each other separately.

Also, it is plain that the same proof may be applied in any case; for at each step the number of forces which are not in pairs is diminished by one; and therefore, the reduction may always be effected by as many steps as there are forces, wanting one.

Hence, the Proposition is manifest; for the pressure upon the fulcrum arising from each pair is equal to the sum of the two forces of that pair (Proposition 2); therefore, the whole pressure is equal to the sum of all the pairs; that is, to the sum of all the forces.

Section 2. Composition and Resolution of Forces. Definitions.

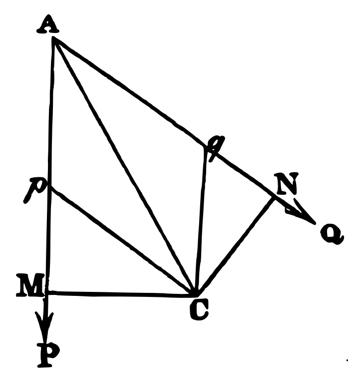
1. When two forces act at the same point, they produce the same statical effect as a certain single force, acting at that point. This single force is called the *resultant* of the two; they are called its *components*. The two forces produce the single force by being *compounded*, and it may be *resolved* into the two.

2. Straight lines may represent forces in direction and magnitude, when they are taken in the direction of the forces and proportional to their magnitude. When forces are so represented if AB represent any force, BA represents an equal and opposite force. A force represented by any line, as AB, is often called “the force AB.”

3. Forces may be *represented* by lines parallel to them in direction and proportional to them in magnitude.

Proposition 8. If the adjacent sides of a parallelogram represent the component forces in direction and magnitude, the diagonal will represent the resultant force ‘in direction and magnitude.

The proof will consist of two parts; for the direction, and for the magnitude.



First, the diagonal will represent the resultant force in *direction.*

Let *Ap*, *Aq* represent in magnitude and direction the forces *P*, *Q*, acting at *A*; complete the parallelogram *ApCq*; and draw *AC*; draw also *CM*, *CN* perpendicular upon *Ap*, *Aq*.

The triangles *CpM*, *CqN* have right angles at *M* and *N*, and the angles *MpC*, *CqN* are equal, each being equal to *MAN*; therefore, the triangles *CpM*, *CqN* are equiangular and similar. Therefore, $CM : CN :: Cp : Cq$; that is, $CM : CN :: Aq : Ap$. But *Ap*, *Aq* represent the forces *P*, *Q*, in magnitude; therefore $CM : CN :: Q : P$. Therefore, by Proposition 6, Corollary: 4, if the forces *P*, *Q* act on the plane *PAQ*, supposed to be moveable about the point *C*, they will balance each other, producing a pressure on the fulcrum *C*.

Therefore, the single force which produces the same effect as *P*, *Q* will produce a pressure upon the point *C*, but will not turn the plane about *C*. But this cannot be the case except the single force act in the line *AC*; for if it acted in any other direction, a perpendicular might be drawn from *C* upon the direction, and the force would produce motion, by Axiom 2. Therefore, the resultant acts in the direction *AC*.

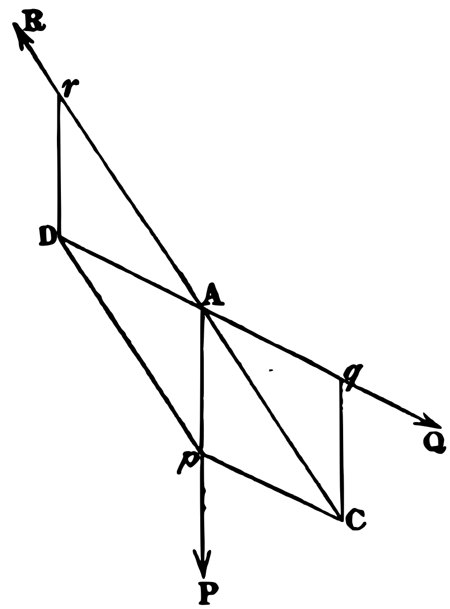
Hence if a point, acted upon by two forces *Ap*, *Aq*, be kept at rest by a third force, this force must act in the direction *CA*. For otherwise it would not balance the force in the direction *AC*, to which the forces *Ap*, *Aq* are equivalent.

Hence also if three forces act on a point, and keep each other in equilibrium, each of them is in the direction of the diagonal of the parallelogram whose sides represent the other two.

Secondly, the diagonal will represent the resultant force in *magnitude*.

By the proof of the former part the two forces *Ap*, *Aq* will be kept in equilibrium by a force in the direction *CA*. Let *Ar* represent this force in magnitude. Therefore, the three forces *Ap*, *Aq*, *Ar* keep each other in equilibrium. ‘Complete the parallelogram *ApDr*, and draw its diagonal *DA*. Then by the proof of the former part, the force *Aq* is in the direction *DA*; and therefore, *DAq* is a straight line.

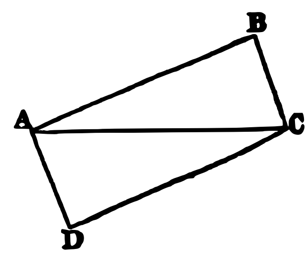
Hence in the triangles *CAq*, *DAr*, the vertical angles *CAq*, *DAr* are equal; and *Cq*, *Dr* are parallel to each other, because *Cq* and *Dr* are both parallel to *Ap*; and *Cr* meets them; therefore, the angle *qCA* is equal to the alternate angle *DrA*. Therefore, the triangles *CAq*, *DAr* are equiangular. Also, *Cq* and *Dr* are equal, for each is equal to *Ap*, being opposite sides of parallelograms *pq*, *pr*. Therefore, (Euclid. 6. 8) the other sides of the triangles *CAq*, *DAr* are equal; therefore, *CA* is equal to *Ar*. But *Ar* represents in magnitude the force which keeps in equilibrium *Ap*, *Aq*; and since *Ar* acting in the opposite direction would balance *Ar*, the force which produces the same effect as *Ap*, *Aq*, is *Ar* acting in the opposite direction. Therefore *AC*, which is equal to *Ar*, represents in magnitude the force which produces the same effect as *Ap*, *Aq*; that is the resultant of *Ap*, *Aq*.



Hence, if the components be represented in magnitude and direction by the sides of a parallelogram, the resultant is represented in magnitude and direction by the diagonal of the parallelogram, Q. E. D.

Proposition, 9. If three forces represented in magnitude and direction by the sides of a triangle taken in order, act on a point, they will keep it in equilibrium.

Let three forces, represented in magnitude and direction by the three fines *AB*, *BC*, *CA*, act on the point *A*, they will keep it in equilibrium. Complete the parallelogram *ABCD*, then the force which is represented by *BC* is also represented by *AD*, (Definition 3 of this Sect.) and acts at the point *A*. And the resultant of the forces *AB*, *AD* is represented in magnitude and direction by *AC* (Proposition 8); therefore, the forces *AB*, *BC* produce the same effect as *AC*; and therefore, the forces *AB*, *BC*, *CA* produce the same effect as *AC*, *CA*; that is, they will keep the point *A* in equilibrium.

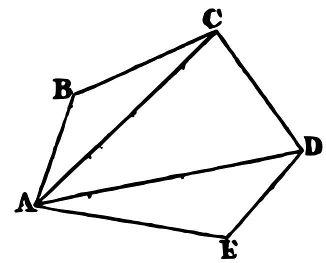


Corollary: 1. If three forces which keep a point in equilibrium be in the direction of three lines forming a triangle, they are proportional to those lines.

Corollary: 2. Any two forces *AB*, *BC*, which act at a point *A*, are equivalent to a force *AC*.

Proposition D. If any number of forces, represented in magnitude and direction by the sides of a polygon taken in order, act on a point, they will keep it in equilibrium.

Let forces *AB*, *BC*, *CD*, *DE*, *EA* act upon a point *A*; they will keep it in equilibrium. By Proposition 9, Corollary: 2, the forces *AB*, *BC* are equivalent to a force *AC*; therefore, the forces *AB*, *BC*, *CD* are equivalent to the forces *AC*, *CD*; that is, by the same corollary, to a force *AD*. Therefore again, the forces *AB*, *BC*, *CD*, *DE* are equivalent to the forces *AD*, *DE*; that is, again by the same corollary, to a force *AE*. Therefore, finally, the forces *AB*, *BE*, *CD*, *DE*, *EA* are equivalent to forces *AE*, *EA*, and therefore will keep the point *A* in equilibrium.



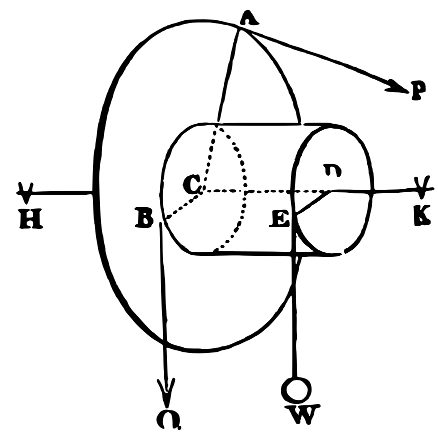
Note JC. How can equilibrium being a comparison between things, suddenly become a judgment about a point, which is not a thing at all? How can one possibly say, that a point is in equilibrium unless they are talking nonsense? How can this have been studied and taught for a long time, not make anyone realize that they were suddenly floating in the ozone layer of gibberish? Is a noun a verb?

Section 3. Mechanical Powers. The Wheel and Axle.

Definition The *Wheel and Axle* is a rigid machine, which is moveable about an axis, and on which two forces, tending to turn it opposite ways, act in two planes perpendicular to the axis; the one force (the Power) acting by means of a string stretched and wrapt on the circumference of a circle perpendicular to the axis, called the Wheel; the other force (the Weight) acting by means of a string wrapt on the surface of a cylinder having the axis of motion for its axis, and called the Axle.

Proposition 10. There is an equilibrium upon the wheel and axle, when the power is to the weight as the radius of the axle to the radius of the wheel.

Let AB be the wheel, and DEB the axle, the whole being moveable about the axis $HCDK$; the power P, acting at A, perpendicular to CA, the radius of the wheel; and the weight W, acting at E, perpendicular to DE, the radius of the axle. Also let $P : W :: DE : CA$; then there will be an equilibrium.



In the plane of the wheel AB, let CB be drawn from the axis, equal to DE the radius of the axle; and let a force Q, equal to W, act at B perpendicular to CB. Then, by Axiom 8, the two forces Q, W produce equal effects in turning the machine. But the force Q will balance P, by Proposition 6, Corollary: 1, because

$P : W :: DE : CJ$, and therefore $P : Q :: CB : CA$,

Q being equal to W, and CB to DE therefore W will balance P, and there will be an equilibrium, Q. E. D.

Corollary: 1. On the wheel and axle when there is equilibrium, the moments of the power and weight are equal.

Corollary: 2. If the power and weight do not act perpendicularly to the radii of the wheel and axle, it will appear, by the reasoning of Proposition 6, that there will still be an equilibrium if their moments are equal.

Corollary: 3. If several forces acting upon a body moveable about a fixed axis, and acting in planes perpendicular to the axis, tend to turn it opposite ways, there will be an equilibrium when the sum of the moments of the forces which tend to turn the body one way is equal to the sum of the moments of the forces which tend to turn the body the other way. This may be proved by reasoning similar to that of Proposition B.

Corollary: 4. If a heavy body be moveable about any axis, it will be in equilibrium when the moments of the weights of the two parts into which it is divided by a vertical plane passing through the axis, are equal: for these two parts will tend to turn it opposite ways.

In this case, the moment of each particle of the body is found by drawing from the particle a vertical line meeting a horizontal line which is perpendicular to the axis. The length of this perpendicular, measured from the vertical to the axis, multiplied into the weight of the particle, is the moment of the particle, if the axis is horizontal; and is proportional to the moment if the axis be in any other position.

Corollary: 5. Conversely, if these moments are not equal, there cannot be equilibrium.

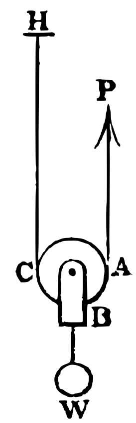
The Pulley.

Definition A *Pulley* is a machine in which one part, (the *Block*) being stationary, a stretched string can pass freely round another part, (the *Sheave*).

A pulley is *fixed* when the block is fixed, and *moveable* when the block is moveable.

The Power is the force which acts at the string; the Weight is the weight supported.

Proposition 11. In the single moveable pulley, where the strings are parallel, there is an equilibrium when the power is to the weight as 1 to 2.

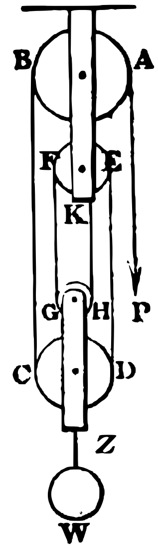


Let *ABC* represent a pulley in which B is the block, *AC* the sheave, and in which the strings *PA*, *HC* are parallel: there is an equilibrium when $P : W :: 1 : 2$.

By Axiom 9, since the string passes freely round the sheave *AC*, the force *P*, which is exerted on the string *PA*, is equal to that which the string *CH* exerts on the fixed-point *H*; and therefore, the reaction which the fixed-point *H* exerts by means of the string *HC*, is also equal to *P*. And the two forces, each equal to *P*, which act by means of the parallel strings *AP*, *CH*, may be considered as balancing each other upon a lever *AC*, the fulcrum of which is in the point of the block *B*, by which the weight *W* is supported. Therefore, by Proposition 2, the pressure on the fulcrum is the sum of these forces, that is, it is the double of *P*; and this pressure on the fulcrum of the block *B* is balanced by the pressure or weight of *W* upon the block in the opposite direction, in the case of equilibrium; therefore, in the case of equilibrium, *W* is double of *P*, or $P : W :: 1 : 2$.

Proposition 12. In a system in which the same string passes round any number of pulleys, and the parts of it between the pulleys are parallel, there is an equilibrium when power (*P*) : weight (*W*) :: 1 : the number of strings at the lower block.

Let AC represent the system of pulleys; the string $A\ B\ C\ D\ E\ F\ G\ H\ K$ passing round all the pulleys, and the portions CB, DE, $GF\ HK$, being all parallel. By Axiom 9, the forces exerted by each of these strings will be equal to P; therefore, the forces which they exert upon the lower block will each be equal to P. And these forces may be considered as acting upon a lever, the fulcrum of which is in the point of the block Z, by which the weight W is supported. Therefore, by Proposition C, the pressure upon this fulcrum is equal to the sum of the forces of the strigs, that is, it is as many times P as there are strings at the lower block. And this pressure on the fulcrum in the lower block is balanced by the pressure or weight of W in the opposite direction in the case of equilibrium; therefore, in the case of equilibrium, $P : W :: 1$: *number of strings in the lower block*. Q. E. D.



Proposition 13. In a system in which each pulley hangs by a separate string, and the strings are parallel, there is an equilibrium when $P : W :: 1$: *that power of* 2 *whose index is the number of moveable pulleys.*

Let AL represent the system of pulleys; each pulley A, C, E hanging by a separate string, and the strings being all parallel. It appears by the reasoning of Proposition 11, that

$$P : \text{force of string } BC :: 1 : 2;$$

force of string BC : force of string DE :: 1 : 2;

force of string DE : force of string FW :: 1 : 2.

And there will be as many such proportions as there are moveable pulleys A, C, E. Also, in compounding these proportions the proportion compounded of the former ratios in each proportion will be P : force of string FW; and the proportion compounded of the latter ratios in each proportion will be 1 : 2 raised to that power whose index is the number of ratios. Therefore

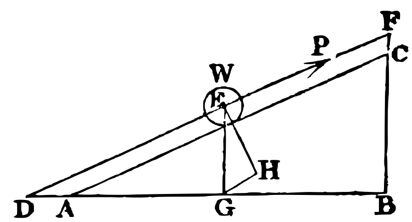
$$P : \text{force of string } FW :: 1 : 2 \text{ raised to that power.}$$

And the force of the string FW is equal to the weight W, because it supports it in the case of equilibrium. Therefore, etc. Q. E. D.

The Inclined Plane.

Definition The *Inclined Plane*, when spoken of as a mechanical power, is a plane supposed to be perfectly smooth and hard. The inclined plane is represented by a line drawn in a vertical plane, and is supposed to pass through this line and to be perpendicular to the vertical plane. A vertical line is supposed to be drawn in the vertical plane from the upper extremity of the inclined plane; and both this vertical line, and the line which represents the inclined plane, are cut by a horizontal line or base, drawn in the same vertical plane. The portion of the inclined line and of the vertical line intercepted between the upper point of the plane and its horizontal base, are the length and the height of the inclined plane respectively.

Proposition 14. The weight (W) being on an inclined plane, and the force (P) acting parallel to the plane, there is an equilibrium when $P : W ::$ the height of the plane : its length.



Let AC be an inclined plane of which AC is the length, and let W be a weight on the inclined plane supported by a force P, acting in the direction EF parallel to AC.

The force of the weight W acts in a vertical direction; draw EG vertical to represent this force. Also draw EH perpendicular and GH parallel to the plane AC.

The force EG is equivalent to the two forces EH, HG, (Proposition 9, Corollary: 2); of these, the force EH is balanced by the reaction of the plane AC, which will balance any force perpendicular to AC, by Axiom 12; and the weight W will be kept at rest, if the force HG be counteracted by an equal and opposite force P, acting in the direction EF. Therefore, there will be equilibrium if P be represented by GH, when W is represented by EG; that is, $P : W :: GH : EG$.

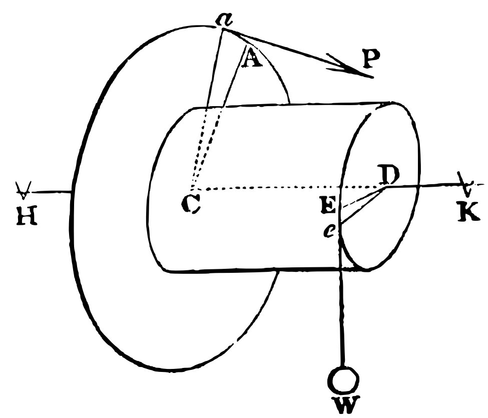
But since EH is perpendicular and GH parallel to the plane AC, EHG is a right angle and therefore equal to ABC. Also, the angle EGH is, by parallels, equal to GED, that is, to BFD, that is, to BCA. Therefore, the two triangles ABC, EHG, have two angles equal, each to each, and are therefore equiangular, and therefore also similar. Hence $GH : EG :: BC : AC$, and therefore, by what has been proved already, $P : W :: BC : AC$, that is, $P : W ::$ height of plane : length of plane. Therefore, etc. Q. E. D.

Velocity.

Definition. If two points pass through certain distances respectively in the same time, the *Velocities* of the two points are to each other in the proportion of these two distances.

Note JC. Can a point, which is not a thing, have a velocity, or is this actually shorthand for a ratio? Does using the shorthand lead to further errors in conception. Yes.

Proposition 15. If P and W balance each other on the wheel and axle, and the whole be put in motion, $P : W :: W$'s velocity : P's velocity.



The construction being the same as in Proposition 10, let the machine turn round its axle *CD* through an angle *ACa*, or *Ede*; so that the radius of the wheel at which the power acted, moves out of the position *Ca*, into the position *CA*; and so that the radius of the axle at which the power acted, moves out of the position *De* into the position *DE*. Then the string by which the power P acts will be unwrapt from the portion *aA* of the circumference of the wheel, and therefore P will move through a distance equal to *aA*. Also, in the same time the string at which W acts will be wrapt upon the axle by a distance equal to *eE*, and therefore W will move through a distance equal to *eE*. Therefore, by the definition of velocity, *aA*, *eE* are as the velocities of P and W.

But since the wheel and axle is a rigid body, turning about the axis *CD*, all the parts move in planes perpendicular to the axis, and turn through the same angles and since the plane of the wheel *ACa*, and of the axle *EDe* are both perpendicular to the axis, the angles *ACa*, *EDe* are the angles through which the radii *CA*, *DE* turn. Therefore, the angles *ACa*, *EDe*, at the centers of the circles *ACa*, *EDe* are equal; and therefore, by the Lemma 3, $DE : CA :: Ee : Aa$.

But by Proposition 10, $DE : CA :: P : W$; and by what has been just shown, $Ee : Aa :: W$'s velocity : P's velocity; therefore $P : W :: W$'s velocity : P's velocity, Q. E. D.

Proposition 16. To show that if P and W balance each other in the machines described in Propositions 11, 12, 13, and 14, and the whole be put in motion, $P : W :: W$'s velocity in the direction of gravity : P's velocity.

Part first: proof for the systems of pulleys described in Propositions 11, 12, 13.

In Proposition 11, if W be raised by any distance, as one inch, the string on each side of the pulley A will be liberated for one inch; and therefore, P will be at liberty to descend two inches; therefore, W's velocity : P's velocity :: 1 : 2; and since by Proposition 11, $P : W :: 1 : 2$, $P : W :: W$'s velocity : P's velocity.

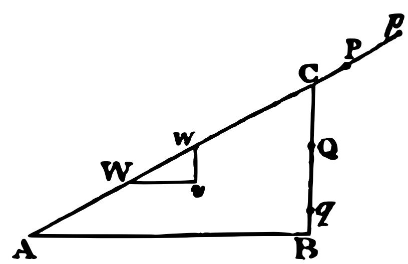
In Proposition 12, if W be raised any distance, as one inch, each string at the lower block will be liberated one inch, and, therefore as many inches of string will be liberated as there are strings at the lower block; and P will be at liberty to descend a distance equal to the whole of this. Therefore, the distance described by W : distance described by P :: 1 : number of strings at the lower block; and hence

by Proposition 12, and by the definition of velocity, P : W :: W's velocity : P's velocity.

In Proposition 13, if W be raised by any distance, as one inch, each of the two strings at the lowest pulley E will be liberated one inch; therefore the pulley C will be liberated 2 inches, and will rise by 2 inches; therefore on each side the block C, 2 inches of string will be liberated; therefore the pulley A will be liberated 2×2 inches; therefore the string on each side the pulley A will be liberated 2×2 inches; therefore the string at which P acts will be liberated $2 \times 2 \times 2$ inches, and since this happens in the same time that W is liberated one inch, W's velocity : P's velocity :: $1 : 2 \times 2 \times 2$. And it is clear that the last term is that power of 2 whose index is the number of moveable pulleys.

But by Proposition 13, P : W :: $1 : 2 \times 2 \times 2$ as before; therefore, by what has been proved, P : W :: W's velocity : P's velocity.

Part second : proof for the Inclined Plane described in Proposition 14.



Let AC be the inclined plane, the weight W being supported by the force P acting parallel to the plane. Let W move to w, and P to p in the same time; and draw Wv horizontal and wv vertical. Then wv is the distance which W moves in the direction of gravity, while P moves the distance Pp, or Ww, which is equal to Pp, because the string wP is always of the same length. Therefore, by the definition of velocity, W's velocity in the direction of gravity : P's velocity :: wv : Ww.

But since Wv is horizontal, or parallel to AB, and wv vertical, or parallel to CB, the triangle Wwv is similar to ACB. Therefore wv : Ww :: BC : AC, that is, wv : Ww :: height of the plane : length of the plane. But by Proposition 14, this proportion is that of P : W; therefore, by what has been proved, P : W :: W's velocity in the direction of gravity : P's velocity.

Corollary: In the case of the inclined plane, if the string by which W is supported, pass over a point C and hang vertically, as WCQ, and if Q balance W, Q will descend through a distance Qq equal to Ww, when W descends through a distance Ww; and we may prove, as before, Q : W :: W's velocity in the direction of gravity : P's velocity.

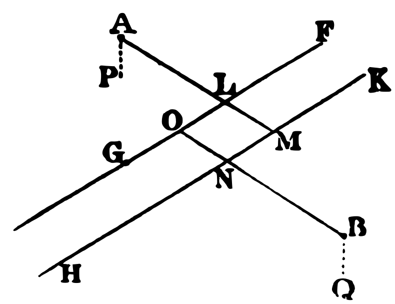
Section 4. The Center of Gravity.

Definition. The *Center of Gravity* of any body or system of bodies is the point about which the body or the system will balance itself in all positions.

Corollary: If a straight-line pass through the center of gravity of a body, the body will balance itself on this line in all positions. For since the body will balance itself in all positions upon the center of gravity, if this center be supported, the body will be supported in all positions. But if the line passing through the center of gravity be supported, the center will be supported; and therefore, if the line passing through the center of gravity be supported, the body will be supported in all positions; therefore, it will balance itself on this line in all positions.

It is assumed that every-body has a center of gravity.

Proposition 17. If a body balance upon a straight line in all positions, the center of gravity is in that line.



Let *HK* be a line on which the system balances itself in all positions; and since every system has a center of gravity, if possible, let *G*, which is not in *HK*, be the center of gravity.

Let *GF* be drawn parallel to *HK*; then, if any line in the plane *FGHK*, as *LM*, or *ON*, be perpendicular to one of these parallels, it will be perpendicular to the other. Let the body, with these lines, be turned round the line *HK*, till *LM* is horizontal, in which case any other perpendicular, as ON, will also be horizontal. Let *P* be a particle, the vertical line from which meets the horizontal line *ML*, produced, if necessary, in *A*; let *Q* be a particle, the vertical line from which meets the horizontal line *ON* in *B*; and in like manner let vertical lines be drawn from the other particles of the body, meeting horizontal lines which are perpendicular to *FG* and *HK*. Also, let *P*, *Q*, be the weights of particles from which the vertical lines *PA*, *QB* are on opposite sides of the lines *GF*, *HK*.

Since the body balances on the line *HK*, the sum of all such moments as $P \times AM$ on the one side of the line *HK* must be equal to the sum of all such moments as $Q \times BN$ on the other side of the line by Proposition 10, Corollary: 4, And since, by the corollary to the Definition of the center of gravity, the body balances on the line *GF*, the sum of all such moments as $P \times AL$ on the one side of the line *GF* must, for the same reason, be equal to the sum of all such moments as $Q \times BO$ on the other side of the line *GF*.

But when we take the moments of the particles of the body with respect to the line *GF*, instead of *HK*, each of the moments on the side A, as $P \times AM$, is diminished by $P \times LM$, so as to become $P \times AL$; and each of the moments on the side *B*, as $Q \times BN$, is increased by $Q \times NO$, so as to become $Q \times BO$: besides which there are particles, the vertical lines from which fall between the lines *HK*, *GF*, which are on the side *A* of the line *HK*, and on the side *B* of the line *GF*, and of which the moments still further diminish the sum of the moments on the side *A*, and increase the sum of the moments on the side *B*, when we exchange the line *HK* for the line *GF*.

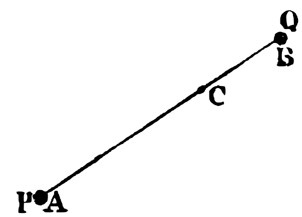
Therefore, if the sums of the moments on the sides A and B of the lines HK be equal, the sums cannot be equal when we move the line into the position GF, and therefore by Proposition 10, Corollary: 5, the equilibrium cannot subsist for this second line also.

Therefore, the point G, out of HK, cannot be the center of gravity; and therefore, the center of gravity must be in HK. Q. E. D.

Proposition 18. To find the center of gravity of two heavy points.

Note JC Our point is picking up more baggage, first they can move, then be in equilibrium, now they can be light and heavy; amazing.

It has been postulated, that in the multiverse theory, that there is at least one of them where there is an intelligent version of the self who has never ripped the fabric of space and time to expose themselves while on a formal date.

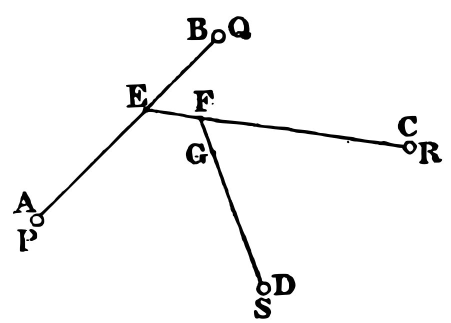


Let A, B, be the two heavy points; their weights being P and Q. Join AB; and take in AB a point C, such that $P + Q : Q :: AB : AC$; C will be the center of gravity of A, B.

Since $P + Q : Q :: AB : AC$, by division $P : Q :: BC : AC$. Therefore, by Proposition 2, A and B will balance each other on the line AB in a horizontal position, because in that case the weights act perpendicularly to the lever. Therefore, by Proposition 7, A, B will balance each other on C in every other position of the line AB. Therefore, by the definition of the center of gravity, C is the center of gravity of the heavy points AB.

Corollary: The pressure upon the center C in every position is equal to $P + Q$, by the Corollary: to Proposition 7.

Proposition 19. To find the center of gravity of any number of heavy points.



Let A, B, C, D be any number of heavy points; their weights being P, Q, R, S. Join AB, and take a point E in AB, such that $P + Q : Q :: AB : AE$; join EC, and take a point F in EC, such that $P + Q + R : R :: EC : EF$; join FD, and take a point G in FD, such that $P + Q + R + S : S :: FD : FG$; G will the center of gravity of P, Q, R, S.

Since $P + Q : Q :: AB : AE$, by Proposition 18, and Corollary: E is the center of gravity of the points A, B; and in every position of AB the, pressure upon E is equal to $P + Q$. But since $P + Q + R : R :: EC : EF$, by division $P + Q : R :: CF : EF$; therefore $P + Q$ at E and R at C will balance upon F when EC is horizontal by Proposition 2, and when EC is in any other position by Proposition 7; and the pressure upon F in any position will be $P + Q + R$, by the Corollary: to Proposition 7. Therefore, in any position P, Q, R will balance upon F, and F is the center of gravity of P, Q, R.

Again, since $P \quad Q + R + S : S :: FD : FG$, by division, $P + Q + R : S :: DG : FG$; and $P + Q + R$ at F, and S at D, will balance in every position of FD, by Propositions 2 and 7. And the pressure upon G will, in every position of FD, be $P + Q + R + S$, by Corollary: to Proposition 7.

Therefore, in every position of *FD*, *EC*, and *BA*, the points *A*, *B*, *C*, *D* will balance upon *G*; and therefore, *G* is the center of gravity of *A*, *B*, *C*, *D*.

Corollary: 1. It has been shown that in every position of *A*, *B*, *C*, *D* the pressure upon *G*, the center of gravity, is equal to the sum of the weights.

Corollary: 2. Every system of heavy points has a center of gravity; for the above construction is always possible.

Proposition 20. To find the center of gravity of a straight line.

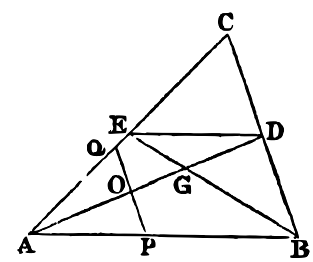
Note JC Is the center of a line the same as the center of gravity? Can a line ever have a center of gravity? Can it possibly have any weight?

Let *AB* be the straight line; bisect it in *C*; *C* will be the center of gravity.

Take *CM* and *CN* equal, and the line may be considered as composed of pairs of equal particles, placed at points such as *M*, *N*, by Axiom 13. But the two particles at *M*, *N* balance each other upon the point C in all positions, by Proposition 2 and 7. And all the other pairs of particles will balance for the like reasons. Therefore, the whole line will balance upon *C* in all positions. Therefore, the point *C* is the center of gravity of the whole line.

Proposition 21. To find the center of gravity of a plane triangle.

Let *ABC* be the triangle; bisect *BC* in *D*, and join *AD*; and bisect *AC* in *E*, and join *BE*; let *G* be the point of intersection of *AD*, *BE*; *G* is the center of gravity of the triangle.



Draw any line *PQ* parallel to *BC*, meeting *AD* in *O*; it is easily seen that the triangles *AOP*, *ADB* are similar, as also *AOQ*, *ADC*.

Hence $OP : OA :: DB : DA$;

and $OA : OQ :: DA : DC$;

therefore $OP : OQ :: DB : DC$.

But *DB* is equal to *DC*, therefore *OP* is equal to *OQ*, and *O* bisects *PQ*.

By Axiom 14, the triangle *ABC* may be considered as made up of straight lines *PQ*, parallel to *BC*. And the center of gravity of any one of these lines, as *PQ*, is at *O* in the line *AD*; therefore, each of these lines will balance upon *AD* in any position; therefore, the whole triangle, which is made up of these lines, will balance upon *AD* in any position, and therefore the center of gravity of the triangle is in the line *AD*.

In like manner, the triangle may be considered as made up of straight lines parallel to *AC*, and it may be proved by similar reasoning that the center of gravity of the triangle is in the line *BE*.

Therefore, the center of gravity of the triangle is at *G*, the intersection of *AD* and *BC*. Q. E. D.

Corollary: If we join *DE*, it is easily shown that the triangles *CBA*, *CDE* are similar as also *AGB*, *DGE*,

therefore $DE : AB :: CD : CB$;

but by construction $CD : CB :: 1 : 2$;

therefore $DE : AB :: 1 : 2$.

Again $GD : AG :: DE : AB$.

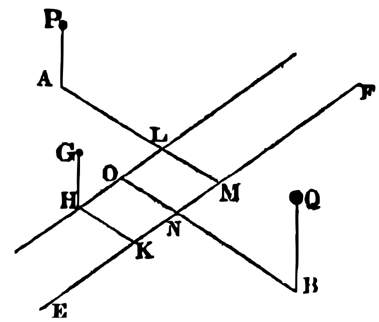
therefore $GD : AG :: 1 : 2$;

and by composition $AD : AG :: 3 : 2$;

AG is two-thirds of AD and DG is one-third of AD.

In like manner BG, and GE, are two-thirds and one-third of BC respectively.

Proposition E. Any body will have the same effect in producing equilibrium about a given fixed line, as if it were collected at its center of gravity.



Let EF be the given fixed line, and G the center of gravity of the body. Let PA, QB vertical lines from any particles P, Q of the body, meet horizontal lines AM, BN, which are perpendicular to EF; and let GH be a vertical line which meets the horizontal line HK which is also perpendicular to EF.

The effect of the body in producing equilibrium depends upon the excess of the moments such as $P \times AM$, on one side of the line EF, above the moments such as $Q \times BN$, on the other side of the line; and is the same so long as this excess is the same. This follows from Proposition 10, Corollary: 3.

Now since G is in the center of gravity, the body balances on the point G, and therefore on the line HL; for if HL be supported, G is supported. Therefore, the sum of all the moments, such as $P \times AL$, on the one side, is equal to the sum of all the moments such as $Q \times BO$, on the other side. And $Q \times BO$ is equal to $Q \times BN + Q \times NO$. Therefore, adding $P \times LM$ to both, the sum of moments such as $P \times AL + P \times LM$, or $P \times AM$, is equal to the sum of moments such as $Q \times BN + Q \times NO + P \times LM$. Therefore, the excess of moments such as $P \times AM$ over moments such as $Q \times BN$ is the sum of moments such as $Q \times NO + P \times LM$; that is, such as $Q \times HK + P \times HK$, or $(Q + P) \times HK$; because LM and NO are each equal HK.

Now if all particles such as P and Q be transferred to G, their effect in producing equilibrium depends upon the sum of moments, such as $(P + Q) \times HK$; therefore, it is the same as before.

Hence if all the particles P, Q be transferred to the center of gravity G, the effect in producing equilibrium is the same as before. But the whole body may be considered as made up of such particles, by Axiom 15. Therefore, if a body be collected at its center of gravity, its effect in producing equilibrium will not be altered. Q. E. D.

Corollary: 1. The effect of the body to disturb equilibrium about a line will be the same as if the body were collected at its center of gravity G. For the effect to disturb equilibrium is the effect to produce equilibrium when an adequate force is applied to counteract the tendency to disturb equilibrium.

Corollary: 2. The effect of a body to produce or disturb equilibrium about a point is the same as if the body were collected at the center of gravity. For any line

being drawn through the point, the effect is the same about this line, by Corollary: 1; and the equilibrium cannot be disturbed about a point, without being disturbed about some line passing through that point.

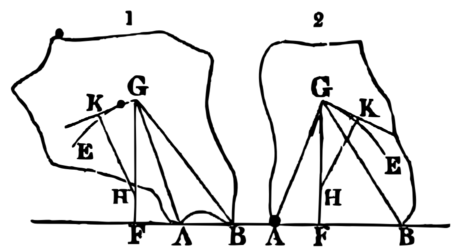
Note. If the fixed line be horizontal, the moment of each particle, which measures its effect in producing equilibrium, is the product of the weight of the particle multiplied by the horizontal line perpendicular to the fixed line and intercepted by a vertical line drawn from the particle.

If the fixed line be not horizontal, take in it any point *Z*, draw *ZY* vertical, and *YX* perpendicular to the fixed line. Then the moment of any particle about the fixed line will be less than the above product in the proportion of *YX* to *YZ*. For the force arising from the weight of the particle being represented by the vertical line *ZY*, may be resolved into forces *ZX*, *XY*; of which *ZX* will not produce any effect to turn the body about the fixed line, and *XY* only will be effective.

Definition By the *Base* of a body is meant a side of it, touching another body, and on which its direct pressure is supported.

If the body fall over, it tends to turn round one edge of its base, whether the base slide or not.

Proposition 22. When a body is placed upon a horizontal plane, it will stand or fall, according as the vertical line, drawn from its center of gravity, falls within or without its base.



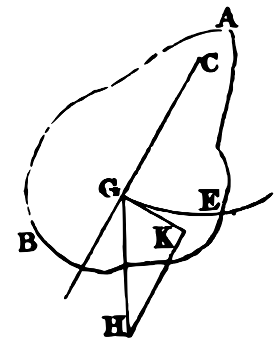
Let *ABCD* be the body, *AB* its base, *G* its center of gravity. First let *GF*, the vertical line drawn from the center of gravity, fall upon the horizontal plane *BA* without the base, as at *F*. Take in *GF* any line *GH* to represent the weight of the body, and draw *GK* perpendicular to *AG* and *HK* parallel to *AG*.

(Fig. 1) If the body fall over the edge *A* of the base, it will tend to turn round the edge *A* of the base, that is, to describe the arc *GE* of which the radius is *AG*. Now by Proposition E, the effect of the body is the same as if it were collected at the point *G*. Therefore, the force exerted to produce this effect may be represented by the vertical line *GH*. And the force *GH* is equivalent to the forces *GK*, and *KH*, (acting at *G*). Of these, the force *KH* acts in the line *GA*, passing through *A*, and therefore produces no tendency to motion about *A*. But the force *GK* tends to make the body move in the direction *GK*, which is a tangent to the arc *GE*; and thus to make the base *AB* turn round the point *A*, quitting the plane at *B*. And there is no force to counteract this tendency; therefore, the body will turn round the edge *A*, on the side on which the perpendicular *GF* falls.

(Fig. 2.) But if the perpendicular *GH* fall between *A* and *B*, as before, the effect may be represented by the vertical line *GH*, and the force *GH* is equivalent to the forces *GK*, *KH*. Of these *KH* (which acts at *G*) passes through *A* and does not tend to make the body turn round the edge *A*; but the force *GK*, which is a tangent to the arc *GE*, tends to make the body turn round *A* in the direction *GE*. But since the body is rigid, and AB is in contact with the supporting plane, the body cannot turn round the point *A* in the direction *GE*, for the pressure thus produced on the

horizontal plane is resisted and supported. In like manner the body cannot turn round the edge *B* by the action of the force *GH*; therefore, in this case the body cannot fall.

Proposition 23. When a body is suspended from a fixed-point, it will rest only with its center of gravity in the vertical line passing through the point of suspension.

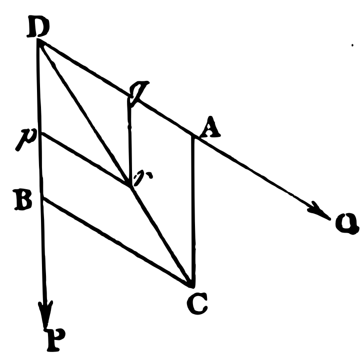


Let *AB* be a body suspended from a fixed-point *C*, and *G* its center of gravity. If *CG* be not vertical, draw *GH* vertical, and (in the vertical plane *CGH*,) *GK* perpendicular to *CG*, and *HK* parallel to *CG*. The weight of the body will produce the same effect as if it were collected at the point *G*, and may be represented by the line *GH*. But the force *GH* is equivalent to *GK*, *KH*; and of these, the force *KH* (which acts at *G*) is in the line *CG*, and is supported by the fixed-point at *C*; and the force *GK* tends to make the body move in *GK*, which is a tangent to *GE*, the path in which the point *G* can move round the fixed-point *C*; and there is no force to counteract this tendency, therefore the body will move in this path; and will not rest in the position *AB*.

But if *CG* be vertical, the weight will be supported by the fixed-point *C*, and there will be no force to produce motion; therefore, the body will rest in that position.

Therefore, the body will rest only when *CG* is vertical, Q. E. D.

Proposition F. If two forces tending to turn a body about a fixed-point, .and acting in a plane perpendicular to the axis of motion, balance each other, the pressure on the fixed-point is the same as it would be if the two forces were transferred to the point retaining their direction and magnitude.



Let *P*, *Q*, be two forces, acting to turn a body about a fixed-point *C*. Draw *CA* parallel to the force *P*, and *CB* parallel to the force *Q*; the pressure on *C* is the same as *P* if the forces *P*, *Q*, acted in the lines *AC*, *BC*.

Produce the directions of the forces to meet in *D*, and complete the parallelogram *CADB*. The force *P* produces the same effect as if it acted at the point *D* in *P*'s direction by Axiom 7; and similarly, the force *Q* produces the same effect as if it acted at *D*. And if *Dp*, *Dq* represent the forces *P*, *Q*, and the parallelogram *Dprq* be completed, the diagonal *Dr* will represent the force at *D* to which *P* and *Q* are equivalent. But the direction of the force *Dr* must pass through the point *C*, as in Proposition 8, and will produce the same effect as if it acted at *C*; and the. force *Dr* acting at *C* is equivalent to the forces *qr*, *pr*, acting in directions parallel to *qr*, *pr*, by Proposition 8: that is, the force *Dr* is equivalent to the forces *Dp*, *Dq*, acting in the lines *AC*, *BC*; that is, the forces *P*, *Q*, acting in the lines *BP*, *AQ* are equivalent to forces *P*, *Q*

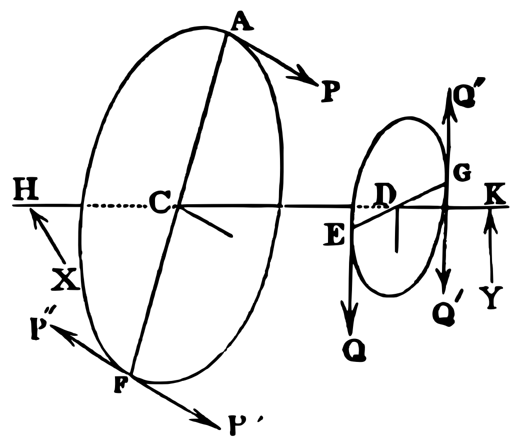
acting in *AC*, *BC*. Therefore, the pressure upon the fixed-point *C* is the same as if the forces *P*, *Q* were transferred to that point. Q. E. D.

Corollary: 1. If, instead of the fixed-point at *C*, we substitute the pressure which that point exerts, there will be equilibrium by Axiom 11. Hence, if a body be acted upon by three forces in the same plane, of which one passes through the intersection of the other two, and is equal to the resultant of the other two, the body will be in equilibrium.

Corollary: 2. Conversely if there be equilibrium, these conditions obtain. This follows from Axiom 2.

Proposition G. If two forces tending to turn a body round a fixed axis, and acting in two planes perpendicular to the axis, balance each other, (as in the Wheel and Axle,) the pressures upon the points of the axis where the body is supported, are the same as they would be, if the two forces, retaining their direction and magnitude, were transferred to the axis, at the points where the perpendicular planes meet it.

Let *P*, *Q*, be two forces acting perpendicularly at the arms *CA*, *DE*, to turn a body round the axis *HK*, the planes *CAP*, *DEQ* being perpendicular to *HK*; and let the forces balance. Let *X*, *Y* be the pressures exerted by the fulcrums at *H* and *K* which pressures balance the forces *P*, *Q*. Then *X* and *Y* are the same as if the forces *P* and *Q*, continuing parallel to themselves, were transferred to *C* and *D*.



Let *AC* be produced to *F*, *CF* being equal to *CA*, and at *F* in the plane *PAC*, and perpendicular to *AF*; let two forces *P*′, *P*″, each equal to *P*, act in opposite directions. These forces will balance each other and will be equivalent to no force; and therefore, if the forces *P*′, *P*″ are added to the system, the equilibrium will not be disturbed.

In like manner produce *ED* to *G*, *DG* being equal to *ED*, and at *G*, in the plane *QED*, and perpendicular to *DG*, let two forces *Q*′, *Q*″, each equal to *Q*, act in opposite directions: these forces will not disturb the equilibrium. Therefore, the six forces *P*, *P*′, *P*″, *Q*, *Q*′, *Q*″, acting in the manner described, will be supported by the forces *X*, *Y*; that is, the eight forces *P*, *P*′, *P*″, *Q*, *Q*′, *Q*″, *X*, *Y*, balance each other.

The forces *P*″, *Q*″, are situated in exactly the same manner with regard to vertical lines and planes drawn upwards, as *P*, *Q* are, with regard to vertical lines and planes drawn downwards. Therefore, *P*′, *Q*″, would balance each other on the axis *HK*, and would produce at *H* and *K* pressures equal and opposite to those which *P*, *Q* produce. But the forces *X*, *Y* are equal and opposite to the pressures which *P*, *Q* produce, for they balance those pressures. Therefore, the forces *P*″, *Q*″ produce at *H*, *K* the pressures *X*, *Y*.

The forces P, P' are equivalent to a force double of P acting at C, parallel to P; and the forces Q, Q' are equivalent to a force at D double of Q, parallel to Q.

Hence, the six forces P, P', P'', Q, Q', Q'' are equivalent to X, Y, at H, K, and to $2P$, $2Q$ at C, D. And the eight forces P, P', P'', Q, Q', Q'', X, Y are equivalent to $2X$, $2Y$ at H, K, and to $2P$, $2Q$, at C, D.

But these eight forces balance each other; therefore $2X$, $2Y$, acting at H, K, balance $2P$, $2Q$, acting at C, D: and therefore, X, Y, which balance P, Q, acting at A, E, would balance P, Q, acting at C, D. Q. E. D.

Book 2. Hydrostatics. Definitions and Fundamental Notions.

1. Hydrostatics is the science which treats of the laws of equilibrium and pressure of fluids.

2. Fluids are bodies the parts of which are moveable amongst each other by very small forces, and which when pressed in one part transmit the pressure to another part.

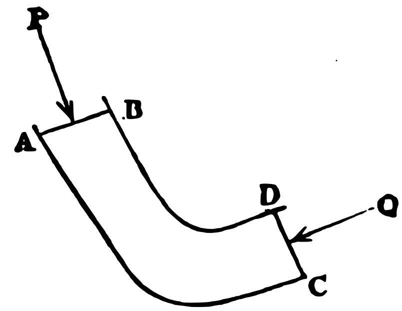
3. Some fluids are *compressible* and *elastic*; that is, they are capable of being made to occupy a smaller space by pressure applied to the boundary within which they are contained, and when thus compressed, they resist the compressing forces and exert an effort to expand themselves into a larger space. Air is such a fluid.

4. Other fluids are *incompressible* and *inelastic*; not admitting of being pressed into a smaller space nor exerting any force to occupy a larger. Water is considered as such a fluid in most hydrostatical reasonings.

5. In all fluids which have weight, the weight of the whole is composed of the sum of the weights of all the parts.

Axioms.

1. If a fluid of which the parts have no weight be contained in a tube of which the two ends are similar and equal planes, two equal pressures applied perpendicularity at the two ends will balance each other.



Note JC is it possible for any fluid to have no weight?

Let *ABCD* be the tube, *AB*, *CD* its two equal ends: the equal forces *P*, *Q*, acting perpendicularly on these ends will balance each other.

Note JC Is it possible to balance anything which has no weight, or is Whewell confused by using the word balance in two different senses as he did with the center of gravity for a line and triangle? Would he then be using the word force, itself, in two different senses? If it has no weight, it cannot possibly react to the force of weight, no more than a blind man can react to the force of sight.

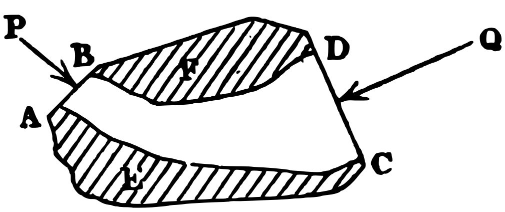
2. If two forces acting upon two portions of the boundary of a fluid balance each other, and if a force be added to one of then it will prevail, and drive out the fluid at the part of the surface acted on by the other force.

Corollary: Hence if *P* and *Q* in Axiom 1 balance, they are equal.

Note JC how often does Whewell form a corollary, proposition, axion; which involves nothing more than synonyms, i.e., inflating a work.

3. If a fluid be at rest in any vessel, and if any forces, acting on two portions of the boundary of the fluid, balance each other, they will also balance each other if any portions of the fluid become rigid without altering the magnitude, position, or weight of any of their parts.

Thus, if the two forces *P*, *Q*, acting on *AB*, *CD*, parts of the surface of a vessel containing fluid, balance each other; they will also balance each other if the parts *E* and *F* of the fluid be supposed to become rigid, the magnitude position and weight, of all the parts of *E*, *F*, regaining unaltered.



4. Any plane surface pressed by a fluid may be divided into any number of particles, and the pressure on the whole is equal to the sum of the pressures on each of the particles.

5. When a plane surface is pressed by a fluid, the pressure exerted on the surface, and the pressure of the surface on the fluid, are perpendicular to the plane.

6. We may reason concerning fluids supposing them to be without weight: and we shall obtain the pressures which exist in heavy fluids, if we add, to the pressures which would take place if the fluids had no weight, the pressures which arise from the weight.

7. When a finite mass of fluid is considered as consisting of small particles of any form or size, and when the consequences of our reasoning do not depend upon

the magnitude of the particles, we may, in our reasoning, neglect the magnitude or weight of any single particle, and the consequences will still be true in a heavy fluid.

Remarks on the Axioms of Hydrostatics.

1. As the Axioms of Geometry are derived from the idea of space, and the Axioms of Statics from the ideas of pressure and of solid coherent matter; the Axioms of Hydrostatics are derived from the idea of pressure, and from the idea of fluid matter;—matter which, without coherence or rigidity, can still sustain pressure and transmit it in all directions; or, as we may express it more briefly, from the idea of fluid pressure. It is not enough to conceive a fluid as a body the parts of which are perfectly moveable: for the mere notion of mobility includes no conception of force or pressure. We must conceive fluid as transmitting pressure, in order to perceive the evidence of the Axioms of Hydrostatics.

2. The First Axiom of our Hydrostatics,—that if a fluid be contained in a tube of which the two ends are similar and equal planes acted on by equal pressures, it will be kept in equilibrium—follows from the principle of sufficient reason, for there is no reason why either pressure should be greater. If, for example, the curvature of the tube or any such cause, affected the pressure at either end, this condition would be a limitation of the property of transmitting pressure in all directions, and would imply imperfect fluidity; whereas the fluidity is supposed to be *perfect*.

3. For the like reasons, we might assume as an *Axiom* the First *Proposition* of the Hydrostatics, that fluids transmit pressure *equally* in all directions, from one part of their boundary to the other; for if the pressure transmitted were different according to the direction, this difference might be deferred to some cohesion or viscosity of the fluid; and the fluidity might be made more perfect, by conceiving the difference removed. Therefore, the proposition would be necessarily and evidently true of a perfect fluid.

4. But instead of laying down this as an axiom, Axiom 3 is introduced—that any part of a fluid which is in equilibrium, may be supposed to become rigid. This axiom leads immediately to Proposition 1, and it is, besides, of great use in all parts of Hydrostatics.

If we had to reason concerning flexible bodies, we might conveniently and properly assume a corresponding axiom for them;—namely, that, of a flexible body which is in equilibrium, any part may be supposed to become rigid. And we might give a reason for this, by saying that rigidity implies forces which resist a tendency to change of form, when any such tendency occurs; but in a body which is in equilibrium, there is no tendency to change of form, and therefore the resisting forces vanish. It is of no consequence what forces *would* act *if there were* a stress

to bend the body: since there *is not* any such stress, the rigidity is not called into play, and therefore it makes no difference whether we suppose it to exist or not.

The same kind of reasons may be given, in order to show, what Axiom 3 asserts, the admissibility of introducing, in the case of equilibrium of a fluid, rigidity, instead of that susceptibility of change of figure, (still greater than flexibility,) which fluidity implies. Since the mass is perfectly fluid, its particles exert no constraint on each other's motions; but then; because they are in equilibrium, no constraint is needed to keep them in their places. They are as steadily kept there (so long as the same forces continue to act) as if they were held by the insurmountable forces which connect the parts of a perfectly rigid body. We may therefore suppose the inoperative forces of rigidity to be present or absent among the particles, without altering the other forces or their relations.* And hence we see the truth of Axiom 3 of the Hydrostatics.

5. The last axiom of Hydrostatics (Axiom 7) is introduced in order that we may be able to reason concerning the quantity of fluid pressure, by supposing the fluid divided into small particles. To speak of the particles as finite would lead us into error, since they are not of any known finite magnitude; and to speak of them as indefinitely small, would involve us in the difficulties of the Higher Geometry, in which the Ideas of Limits or Differentials are introduced. The Axiom will be self-evident if we consider the particles as microscopic in magnitude, and of corresponding weight.

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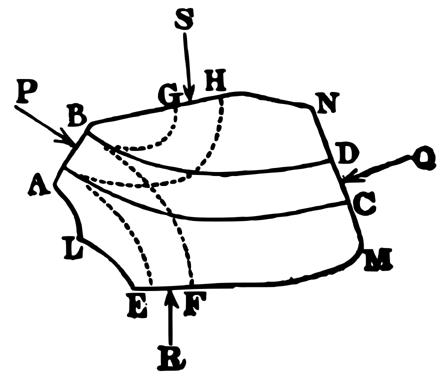
* This Axiom is employed familiarly by Newton and many other eminent mathematicians.

Section 1. Pressure of Non-Elastic Fluids. Propositions.

Proposition 1. Fluids press equally in all directions.

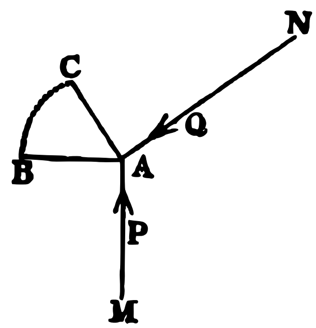
First, a fluid at rest presses equally in all directions on equal plane portions of the vessel which contains it, if we neglect the weight of the fluid.

Let *LMN* be the close vessel, *AB*, *CD*, *EF*, *GH*, similar and equal plane portions of the surface of the vessel; let two forces *P*, *Q* acting on *AB*, *CD*, portions of the boundary of the fluid, balance each other; and let a tube *ACBD* be imagined, passing from *AB* to *CD*. Let the portions of the fluid, *ACL*, *BDN* become rigid; then, by Axiom 3, the forces *P*, *Q* still balance each other; but by Corollary: to Axiom 2, in this case the forces *P*, *Q* are equal. And in like manner it may be shown that the forces *P*, *R* are equal, as also the forces *P*, *S*. And *P*, *Q*, *R*, *S* the forces which act on the boundary of the fluid and balance each other, are the pressures on similar and equal portions of the containing vessel. Therefore, the pressures exerted on all such portions are equal.



Secondly, in a fluid at rest any particle is equally pressed in all directions upon similar and equal plane surfaces.

Let *A* be any point in a fluid, and let *AM*, *AN* be way two directions. Let *AB*, be a plane perpendicular to *AM*, and *AC* a similar and equal plane perpendicular to *AN*. Let the (geometrical) solid, of which the planes *AB*, *AC* are boundaries, be completed, and be considered as a particle of the fluid. And let *P*, *Q* be the forces which act on the planes *AB*, *AC*, and preserve the equilibrium. Let the whole of the fluid which surrounds the solid *ABC* be supposed to become rigid: therefore, by Axiom 3, the forces *P*, *Q* still balance each other.



Let the portion *BAC* of fluid have no weight; therefore, by the proof of the first part, the forces *P*, *Q* are equal to each other.

But by Axiom 7, since this consequence does not depend upon the magnitude of the particle *ABC*, we may neglect the weight of the particle *ABC*, and the consequence will be true.

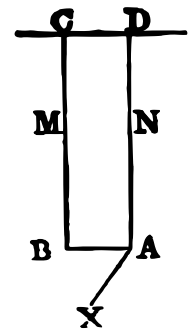
Therefore, in a fluid at rest, the pressures *P*, *Q*, which act upon a particle in the two directions *MA*, *NA*, are equal, Q. E. D.

Corollary. A particle of fluid is equally pressed on any two equal and similar portions of its surface.

Proposition 2. The pressure upon any particle of a [heavy] fluid of uniform density is proportional to its depth below the surface of the fluid.

First, when there is a vertical column of fluid reaching from the particle to the. upper surface.

When the surface pressed is horizontal, let *AB* be the horizontal surface pressed, and *ABCD* the column reaching to the surface, the sides *AD*, *BC* being vertical.

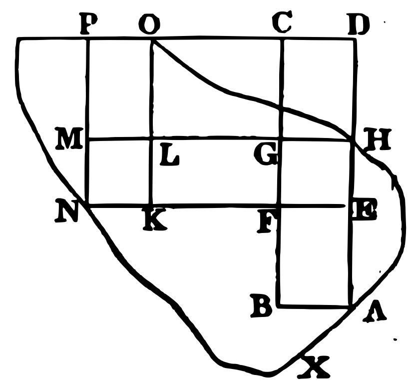


Let the column *ABCD* be divided into any number of equal particles by horizontal planes, drawn at equal vertical intervals. And each of these particles will sustain the pressure of the particle above it, and will transmit this pressure to the particle below it, by Proposition 1; and will also press upon the particle below with its weight, by Axiom 6. Therefore, the pressures on the particles at the distances of 1, 2, 3, etc., intervals below the surface will be as 1, 2, 3, etc.: that is, they will be as the depths.

When the surface pressed is not horizontal, let *AX* be another plane surface of the particle pressed, equal to the plane *AB*. By Proposition 1, the pressure upon *AX* is equal to the pressure upon *AB*; therefore, it is as the depth *AD*.

Secondly, when there is not a vertical column reaching from the particle to the upper surface.

Let *AB* be the horizontal surface pressed, *OP* the surface of the fluid, *OD* horizontal; and *AD* vertical.



Draw *AH* vertical till it meets the side of the vessel; take *HE* = *AB*, and draw *EN* horizontal till it meets the opposite side of the vessel; take *NK* = *AB*, and draw *KO* vertical; and so on if necessary; we shall in this way arrive at the upper surface of the fluid. Draw *HM*, *NP*, so as to complete the zigzag tube *ABEMO* which passes from the plane *AB* to the upper surface of the fluid. Also, the surfaces *GH*, *EF*, *LM*, are all equal to *AB*.

Let the column *MP*, and also the column *BC* be divided into equal particles by horizontal planes at equal vertical intervals, as in the former part of the proof. Then the pressure upon *EF* is equal to the pressure upon *GH*, together with the weight of the particle *GE*, by Axiom 6. But the pressure upon *GH* is equal to the pressure upon *ML*, by Proposition 1, because *GH* is equal to *ML*. Therefore, the pressure upon *EF* is the same as if a column *ED* extended to the surface: and therefore, as in the proof of the former part, the pressure on any particle in *AE* is the same as if a column *AD* extended to the surface; that is, by the former proof, it is as the depth *AD*. Q. E. D.

Corollary: 1. Hence it appears that if a heavy fluid be contained in a vessel of which some parts are over the fluid, any particle of such a part exerts a pressure downward upon the fluid, equal to that which would exist if there were, instead of the particle of the vessel, a vertical column of fluid extending to the horizontal surface of the fluid.

Thus, the particle of the side of the vessel which is over *GH*, presses downwards with the same force as if, instead of that particle of the vessel, there were a vertical column of fluid *GHDC*.

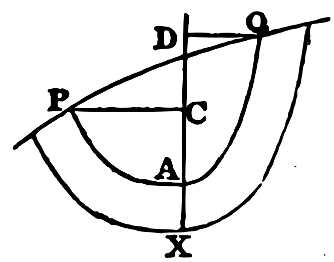
Corollary: 2. Any portion of the side of a vessel which is over the fluid, presses downwards upon the fluid with the same force as if there were a vertical column of fluid over that part, and the side of the vessel were removed.

The part *OH* of the side of the vessel presses downwards with the same force as if the side *OH* were removed, and there were a column of fluid *OHD* over the fluid *OLH*.

For the pressure of the part of the side *OH* downwards is the sum of the pressures of each particle of *OH* downwards; which is, by Corollary: 1, the sum of vertical columns, reaching to the horizontal surface, and standing upon each particle of *OH*: and the sum of these vertical columns, is a column standing on the part *OH*, and reaching to the surface. Therefore, the whole downward pressure is equal to the whole column.

Proposition 3. The upper surface of a heavy fluid of uniform density, and at rest, is horizontal.

Let *PQ* be the upper surface of a heavy fluid. If possible, let *P*, *Q* not be in a horizontal plane. Let *A* be any point in the fluid, *AX* the plane surface of a particle. Draw *PC*, *QD* horizontal, and *ACD* vertical.



By Proposition 2. the pressure upon *AX* arising from the weight of the fluid is as *AC* on the side *P*; and for the same reason it is as *AD* on the side *Q*: and these are opposite pressures upon the plane *AX*. Therefore, the fluid cannot be at rest except these are equal; that is, except *AC* = *AD*; therefore, *PQ* is not otherwise than horizontal. Q. E. D.

Proposition 4. If a vessel, the bottom of which is horizontal, and the sides vertical, contain a heavy fluid, the pressure upon the bottom is equal to the weight of the fluid.

The pressures of the vertical sides are horizontal, and do not increase or diminish the pressure downwards. Therefore, the whole weight of the fluid will be sustained in the same manner as if there were no forces acting on the sides. Let the whole fluid become rigid. Then since it is now a solid (rigid) body, the pressure upon the base is equal to the weight of the body. But by Axiom 3, the pressure is the same as before; therefore, the pressure of the fluid on the base is equal to the weight, Q. E. D.

Corollary: 1. The pressure of a vertical column of height *H* on its horizontal base *B* is as $B \times H$: for this is the content of the column.

Corollary: 2. If *AX* (fig. p. 87) be a particle of the bottom which is not horizontal, the pressure on *AX* is as $AX \times AD$: for if $AB = AX$ be horizontal, the pressure on *AB* is as $AB \times AD$, by Corollary: I: and the pressure on *AX* is equal to the pressure on *AB*, by Proposition 1.

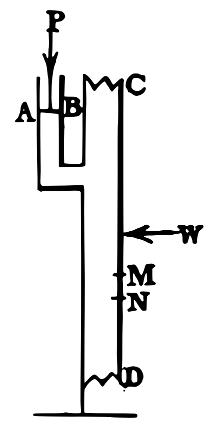
Proposition 5. To construct and explain the hydrostatic paradoxes.

The hydrostatic paradoxes are,

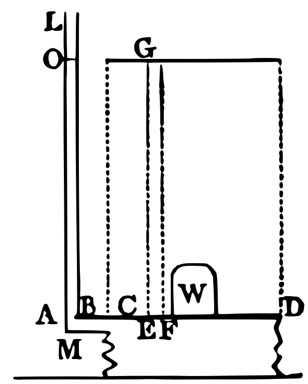
1. That any pressure *P*, however small, may, by means of a fluid, be made to balance any other pressure *W*, however great.

2. That any quantity of fluid, however small, may, by means of its weight, be made to balance a weight *W*, however great.

1. The ratio of *W* to *P*, however great, may be expressed by a number *n*.



Let two planes, *AB*, *CD* be taken, such that 1 : *n* :: *AB* : *CD*; and let a closed machine be constructed in which these planes are moveable, so .as they cart exert pressure on the fluid: as, for example, if *AB* be a *piston*, or plug sliding in a tube, which enters a vessel, and if *CD* be a rigid plane closing a flexible part of the vessel, like the board of a pair of bellows; and let *P* act on *AB*, and let the fluid be in equilibrium. Then the plane *CD* may be divided into n surfaces, each (*MN*) equal to *AB*. By Proposition 1, the pressure upon each of these surfaces is *P*, and hence the whole pressure on *CD* is (Book 1. Proposition C.) the sum of all these pressures: that is, it is *n* times *P*; and if therefore *W* be *n* times *P*, *W* acting at the surface *CD* will be balanced by *P* acting at *AB*.



2. Let the given quantity of fluid be a column of which the base is *B* and the height *H*, and let the given weight *W* be equal to *n* times the weight of this column. Take a plane *CD* equal to n times *B*, and let a machine be constructed in which there is a vertical tube *LM*, of which the horizontal section *AB* is the surface *B*, and which enters a vessel; and let *CD* be a horizontal plane moveably connected with the vessel, as before. And let the vessel *LMND* be filled with the fluid up to the plane *CD*, and let the weight *W* be placed on the plane *CD*, and the tube *LM* be filled with fluid to the point *O* at the height *H* above *CD*, so that *ABCD* being horizontal, *AO* is equal to *H*.

The fluid *BO* and the weight *W* will balance each other.

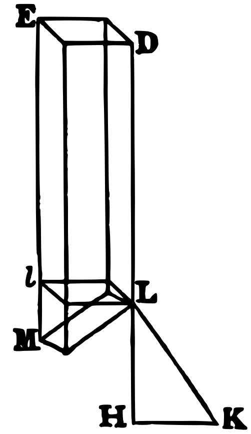
For the plane *CD* may be divided into *n* particle as *EF*, each equal to the plane *B*; and *OG* being horizontal, the pressure of the fluid upwards on each of these is equal to a column of fluid of base *B* and height *AO* or *H*, by Proposition 4. and its Corollaries. Therefore, the whole pressure upwards is *n* times this column. Therefore, if the weight *W* be *n* times this column, the pressures downwards and upwards will balance each other, and there will be an equilibrium.

Proposition 6. If a body floats in a fluid it displaces as much of the fluid as is equal in weight to the body; and it presses downwards and is pressed upwards with a force equal to the weight of fluid displaced.

First, if the fluid be entirely under the body.

Let *LM* be a particle of the surface of the body; and on *LM* let a vertical column be erected, meeting the upper surface of the fluid in *DE*. Draw the horizontal section *Ll* of the column; and take *KL* perpendicular to *LM* to represent the pressure on *LM*, and draw *KH* perpendicular on the vertical line *DL*.

The force KL may be resolved into KH, HL of which HL represents the vertical force; and the whole force on LM is to the vertical force on LM as KL to HL; that is, by Lemma 7, as LM to Ll; or as $DL \times LM$ to $DL \times Ll$. But the whole force on LM is equal to a column of fluid $DL \times LM$, by Corollary: 2. to Proposition 4.; therefore, the vertical force on LM is equal to a column of fluid $DL \times Ll$; that is, to the column $EDLl$, by Lemma 6; that is, to the column $EDLM$, because the single particle LlM may be neglected, by Axiom 7.

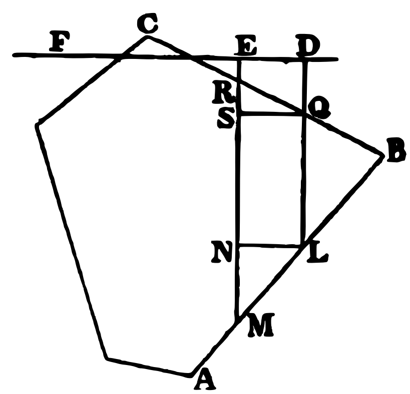


And, in like manner, the vertical pressure upon any other particle of the surface of the floating body is the weight of fluid equal to the vertical column which stands upon that particle, reaching up to the surface of the fluid.

And the whole vertical pressure upwards is equal to the sum of all these columns, that is, to the weight of the fluid displaced.

Secondly, if the fluid be above any part of the body, the excess of the vertical pressures upwards above the vertical pressures downwards is equal to the weight of the fluid displaced.

Let ABC be a vertical section of the body, EF the upper surface of the fluid, LM any particle of one of the lower surfaces of the body.



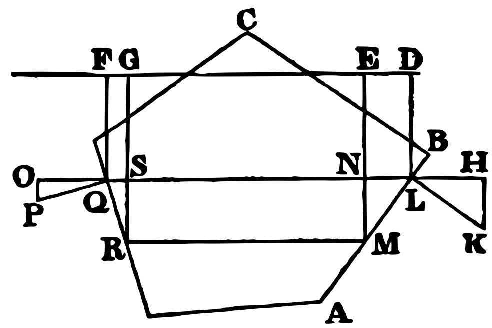
Draw the column $LDME$ vertical, meeting the upper surface of the fluid in DE, and cutting off a particle QR in the upper surface of the body. It may be proved, as in the former part, that the vertical pressure upwards on the particle LM is equal to the weight of the column of fluid $LDEM$. And in the same manner it may be proved that the vertical pressure downwards on the particle QR is equal to the weight of the column of fluid $QDER$. Therefore, the excess of the pressures upwards above the pressures downwards on this vertical column is the excess of the weight of the column of fluid $LDEM$ over that of $QDER$; that is, it is the weight of the column $LQRM$.

In the same manner, in any other vertical column, the excess of the pressure upwards above the pressure downwards is the weight of a quantity of fluid equal to the vertical column intercepted within the body. And the whole excess of the vertical pressures upwards is the sum of all such intercepted columns; that is, it is the weight of the fluid displaced by the body.

Therefore, in all cases, the weight which can be supported by the pressure upwards, or by the excess of the pressure upwards, is the weight of the fluid displaced. But if a body float the weight of the body must be supported. Therefore, the weight of the fluid displaced must be equal to the weight of the body.

And in this case the body presses downwards with its weight, that is, with the weight of the fluid displaced; and it is supported by an equal pressure upwards. Q. E. D.

Proposition A. If any horizontal prism be wholly or partially immersed in a fluid of uniform density, the horizontal pressures of the fluid on the sides of the prism cancel out each other.



Let *ABC* be a vertical section of the prism perpendicular to its length, *EF* the upper surface of the fluid; *LM* any particle of one of the surfaces of the prism. Draw *LQ*, *MR* horizontal, cutting off *QR*, a particle of the opposite surface of the prism. Draw *LD*, *ME*, *QF*, *RG*, vertical, to the upper surface of the fluid.

Since the plane *ABC* is perpendicular to the length of the prism, the pressures on the sides of the prism, which, by Axiom 5, are perpendicular to the sides, are in the plane *ABC*. Take *KL*, perpendicular and equal to *LM*, to represent the pressure on *LM*. and draw *NLH* horizontal, and *KH* vertical.

By Proposition 4. Corollary: 2, the pressures on the particles *LM*, *QR* are as *LM* × *LD* and *QR* × *QF*; that is, as *LM* and *QR*, because *LD* and *QF* are equal. Therefore, if a line *KL*, equal to *LM*, represent the force on *LM*, a line equal to *QR* will represent the force on *QR*. Let, therefore, *PQ*, perpendicular and equal to *QR*, represent the force on *QR*, and draw *SQO* horizontal and *PO* vertical.

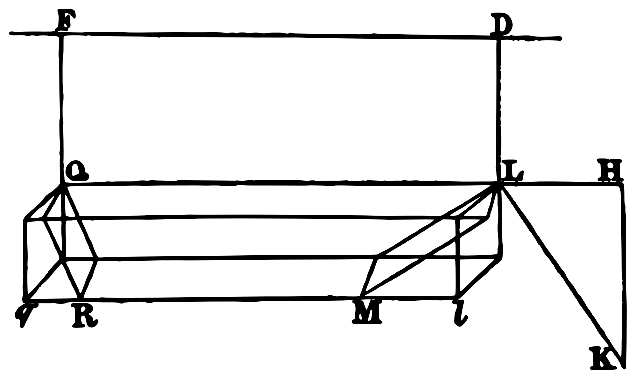
Since *MLK*, *LHK* are right angles, the angles *MLN*, *LKH* are equal: and hence, *LK* being equal to *LM*, the triangles *KHL*, *LNM* are equal in all respects, so that *LH* = *MN* also in like manner the triangles *POQ*, *QSR* are equal in all respects, so that *OQ* = *RS*. But *MN* is = *RS*; therefore *LH* = *OQ*.

The force *KL* may be resolved into *KH*, *HL*, of which *HL* is the horizontal part; and the force *PQ* may be resolved into *PO*, *OQ*: of which *OQ* is the horizontal part; and *OQ*, *HL* have been shown to be equal: therefore, the horizontal forces on the two particles *LM*, *QR* are equal and opposite; therefore, they cancel out each other.

In the same manner, if any other lines be drawn horizontally in the plane of the figure, they will cut off, in the surface of the prism, opposite particles, on which the horizontal forces will cancel out each other; and the horizontal forces on all such particles are the whole horizontal pressures of the fluid on the sides of the prism. Therefore, the whole horizontal pressures cancel out each other, Q. E. D.

Proposition B. If a body bounded by plane surfaces be wholly or partially immersed in a fluid, the horizontal pressures of the fluid on the sides of the body, in any direction and its opposite, cancel out each other.

Let *LM* be a particle of the immersed surface of the body, and on *LM* let a horizontal prism be constituted, (of which *QL*, is one of the edges,) meeting the opposite surface of the body, and cutting off the particle *QR*. Draw *LD*, *QF*, vertical lines, to the upper surface of the fluid. Take *KL* to represent the pressure on *LM*, and draw *KH* perpendicular on *QL* produced. And let *Ll*, *Qq* be the sections of the horizontal column by vertical planes.



The force *KL* may be resolved into *KH*, *HL*, of which *HL* is the horizontal force parallel to the line *LQ*. And the whole force on *LM* is to this horizontal force as *KL* to *HL*; that is, by Lemma 7, as *LM* to *Ll*, or as *LD* × *LM* is to *LD* × *Ll*. But the whole pressure on *LM* is the weight of the column of fluid *LD* × *LM*, by Proposition 4. Corollary: 2. Therefore, the horizontal force on *LM* parallel to *LQ* is the column *LD* × *Ll*.

In like manner, it may be shown that the horizontal force on *QR*, parallel to *QL*, is the weight of the column of fluid *QF* × *Qq*, which is equal to the column *LD* × *Ll*, because, by Lemma 5, *Ll*, *Qq* are equal.

Therefore, the horizontal pressures on *LM* and *QR*, parallel to the line *LQ*, are equal and opposite, and therefore they cancel out each other.

And, in the same manner, the horizontal pressures on any other two opposite particles, parallel to the line *LQ*, cancel each other. And the sum of all such horizontal pressures on opposite particles is the whole pressure on the surface of the body parallel to *LQ*. Therefore, the whole of the horizontal pressures parallel to *LQ* cancel each other.

And, in like manner the whole of the horizontal pressures parallel to any other horizontal line cancel each other.

Therefore, the whole of the horizontal pressures in any direction and ats opposite cancel out each other. Q. E. D.

SCHOLIUM.

The last two Propositions are true of bodies bounded by curvilinear, as well as by plane surfaces. For the curvilinear figure is the limit of a polyhedral figure of a great number of sides. And what is true up to the limit is true of the limit.

Proposition C. When a body floats in a fluid, the centers of gravity of the body and of the fluid displaced are in the same vertical line.

When a body floats, its weight is balanced by the vertical pressures of the fluid on each particle of the immersed surface of the body; and these latter pressures, by Proposition 6., are equal to the weight of vertical columns which would make up the fluid displaced. And the weights of these vertical columns will produce the

same effect, as if they were collected at their center of gravity, and acted upwards there, (Book 1. Proposition E.), that is, at the center of gravity of the fluid displaced. And the weight of the body produces the same effect as if it were collected at its center of gravity, and acted downwards there. Therefore, the two equal forces, one acting vertically upwards at the center of gravity of the fluid displaced, and the other acting vertically downwards at the center of gravity of the body, balance each other. But this cannot be, except they act in the same vertical line; therefore, the two centers of gravity are in the same vertical line, Q. E. D.

Section 2. Specific Gravities.

Definition 1. The *Specific Gravity* of a substance is the proportion of the weight of any magnitude of that substance to the weight of the same magnitude of a certain standard substance (pure water).

For example, if a cubic foot of stone be three times as heavy as a cubic foot of pure water, the specific gravity of the stone is 3.

Definition 2. The *density* is as the quantity of matter in a given magnitude, (Book 1. Article 13), and the quantity of matter is conceived to be as the weight: therefore, the density of a body is as the specific gravity.

Definition 3. When a body lighter than water is entirely immersed in water, it tends to ascend by a certain force which is called its *levity*.

Proposition 7. If M be the magnitude of a body, S its specific gravity, and W its weight, W varies as MS.

If the specific gravity increases in any ratio, the weight of a given magnitude increases in the same ratio, by the Definition of specific gravity; that is, the weight W varies as the specific gravity S; also, if the specific gravity be given, the weight W increases as the magnitude M; therefore, by the Introduction, Article 57, if neither S nor M be given, W varies as MS.

Corollary: If A be the weight of a unit of magnitude of the standard substance (pure water), $W = AMS$.

For W is equal to MS with some multiplier, whole or fractional, by the Proposition. And when M is 1, and S is 1, by supposition $W = A$; therefore $W = AMS$ in all cases.

SCHOLIUM

The weight of a cubic foot of water (A) is 63 pounds avoirdupois nearly.

The following is a list of the specific gravity of various substances; the standard (1) being pure water:—

Gold	19.3
Mercury	13.6
Lead	11.3
Silver	10.5
Copper	8.9
Iron	7.3
Marble	2.7
Water	1.0
Oak	1.2
Fir	.50
Cork	.21
Air	.00125 or $\frac{1}{800}$

EXAMPLES.

1. To find the weight of a cubic inch of silver.

The formula $W = AMS$ being applied in this case,

A is 63 pounds, M is 1 inch cubed, or $\frac{1}{\textit{foot cubed}}$, or $\frac{1}{1728}$ foot S is 10.5;

$$\text{whence } W = \frac{63 \times 10.5}{1728} \text{ pounds} = \frac{16 \times 661.5}{1728} \text{ ounces} = 6.1 \text{ ounces.}$$

2. To find the weight of 10 feet square of gold leaf one thousandth of an inch thick.

$$M = 100 \times \frac{1}{12000}, \ S = 19.3, \ W = \frac{63 \times 19.3}{120} = 10.1 \text{ pounds.}$$

3. To find the weight of a cubical block of marble 1000 feet in the side.

$$W = 63 \times 1000^3 \times 2.7 = 130100000000 \text{ pounds} = 58531250 \text{ tons.}$$

4. To find the weight of a column of air one inch base and 5 miles high.

$$W = 63 \times \frac{5 \times 5280}{144} \times .00125 = \frac{63 \times 5 \times 110}{3 \times 800} = 14 \text{ pounds.}$$

Proposition 8. When a body of uniform density floats on a fluid, the part immersed is to the whole body as the specific gravity of the body is to the specific gravity of the fluid.

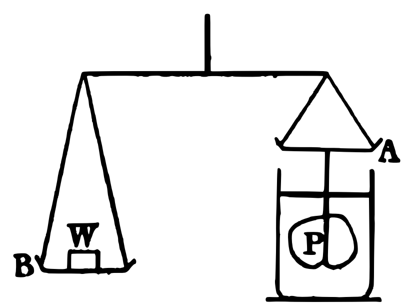
For the magnitude of the part immersed is to that of the whole body as the fluid equal in bulk to the part immersed is to the fluid equal in bulk to the whole body. But the fluid equal in bulk to the part immersed is equal in weight to the whole body, by Proposition 6. Therefore, the part immersed is to the whole as the weight of the body is to the weight of an equal bulk of fluid; that is, by the Definition of specific gravity, as the specific gravity of the body to that of the fluid, Q. E. D.

Proposition 9. When a body is immersed in a fluid, the weight lost in the fluid is to the whole weight of the body as the specific gravity of the fluid is to the specific gravity of the body.

When the body is wholly immersed, the pressure of the fluid vertically upwards is equal to the weight of a bulk of fluid equal to the body, by Proposition 6. But this pressure up wards diminishes the weight of the body when it is immersed in the fluid, and so much weight is lost. Therefore, the weight lost in the fluid is equal to the weight of a bulk of fluid equal to the body. And the specific gravity of the fluid is to the specific gravity of the body, as the weight of a bulk of fluid equal to the body is to the weight of the body (Definition); that is, as the weight lost is to the whole weight. Q. E. D.

Proposition 10. To describe the hydrostatic balance, and its use in finding the specific gravity of a body.

First, when the body is heavier than the fluid in which it is weighed (water).



The hydrostatic balance is a balance in which a body (*P*) can be weighed, either out of water, in the scale *A*, in the usual manner, or in the water (as at P).

In order to find the specific gravity of any body, let it be weighed out of water, and in water; the difference is the weight lost in water; and hence the specific gravity is known by the last Proposition.

Corollary: If U be the weight of the body out of water, V the weight in water, W the weight of an equal bulk of water, and S the specific gravity,

$$W = U - V, \text{ and } S = \frac{U}{W} - \frac{U}{U - V}.$$

Secondly, when the body is lighter than water.

Let the proposed body be weighed out of water; also let it be fastened to a sinker of which the weight in water is known, and let the compound body be weighed in water.

The excess of the weight in water of the sinker, above the weight in water of the compound body, is the levity of the proposed body: for by attaching the proposed

body, its levity or tendency upwards in water diminishes the weight in water of the sinker.

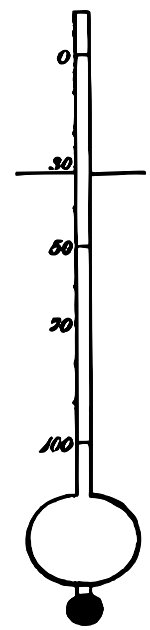
The levity of the proposed body, together with its weight out of water, are equal to the weight of an equal bulk of fluid; for the levity of the body in water is the excess of the pressure upwards above the pressure downwards; that is, the excess of weight of an equal bulk of fluid above the weight (out of water) of the body.

Hence the weight of an equal bulk of water is known, and hence the specific gravity, by the Definition of specific gravity.

Corollary: If U be the weight of the body out of water, Q the weight of the sinker in water, and R the weight of the compound body in water. The levity of the proposed body is $Q - R$. Hence $Q - R + U$ is the weight of an equal bulk of fluid; and

$$S = \frac{U}{Q - R + U}.$$

Example. The weight of the body is 2, and by attaching to it a sinker which weighs 4 in water, the compound body weighs 3 in water. Therefore, the levity of the body is 4 − 3 or 1, and the weight of an equal bulk of fluid is 2 + 1 or 3. Hence the specific gravity is ⅔.



Proposition 11. To describe the common hydrometer, and to show how to compare the specific gravities of two fluids by means of it.

The common Hydrometer is an instrument consisting of a body and a slender stem, and of such specific gravity that in the fluids, for which it is to be used, it floats with the body wholly immersed and the stem partially immersed.

The part immersed is to the whole as the specific gravity of the body is to the specific gravity of the fluid (Proposition 8); and if the specific gravity of the fluid vary, the part immersed will vary in the inverse ratio of the specific gravity.

But since the stem is slender, small variations of the part immersed will occupy a considerable space in the stem, and will be very easily ascertained.

If the magnitude of the whole instrument be represented by 4000 parts and each of the divisions of the stem by 1 such part; and if the whole length of the stem contains 100 such parts, the instrument will measure with great accuracy specific gravities of fluids within certain limits.

Let the fluids be compared with a certain “proof” standard, as 50, in the middle of the scale. If the instrument sink to 30, the specific gravity of the fluid is known. For the part immersed is 4000 − 30, or 3970: and the “proof” fluid, the part immersed is 4000 − 50, or 3950. Therefore, the specific gravity of the fluid is to that of “proof fluid” as 3950 to 3970, or as 395 to 397.

Section 3. Elastic Fluids.

Proposition D. *Inductive Principle* 1 Water and other liquids have weight in all situations.

The facts included in this induction are such as the following-—

(1). Water falls in air as solid bodies do.

(2). A bucket of water held in air is heavy and requires to be supported in the same as a solid body.

(3) A bucket of water held in water appears less heavy than in air, and may be immersed so far as not to appear heavy at all.

(4). A lighter liquid remains, at rest above a heavier, as oil of turpentine upon water.

(5). The bodies of diverse plants, and other organized bodies, though soft, are not compressed or injured under a considerable depth of water.

The different effects (2) and (3) led to the doctrine that all the elements have their proper places, the place of earth and heavy solids being lowest, of heavy fluids next above, of light fluids next, of air next; and that the elements do not gravitate when they are in their proper places, as water in water; but that water in air, being out of its proper place, gravitates, or is heavy. In this way also (1) and (4) were explained.

But it was found that this explanation was not capable of being made satisfactory; for—(6) a solid body of the same size and weight as the bucket of water in (3) gave rise to the same results; and these could not be explained by saying that the solid body was in its proper place.

These facts can be distinctly explained and rigorously deduced, by introducing the *Idea* of *Fluid Pressure*; and the *Principle* that water is a heavy fluid, its weight producing effect according to the laws of fluid pressure.

For on this supposition (1) and (2) are explained, because water is heavy; and (3) is explained by the pressure of the fluid upwards against the bucket, according to Propositions 1, 2, 4.

Also, it may be shown by experiment that in such a case as (1) the lighter fluid increases the pressure which is inserted in the lower fluid.

Facts of the nature of (5) are explained by considering than an equal pressure is exerted on all parts of the organized structure, in opposite directions; such pressures balance each other, and no injury results to the structure, except in some cases a general contraction of dimensions. If there be a communication between the fluids which are within the structure and the fluid in which it is placed, these pressures are exerted from within as well as from without, and the balance is still more complete.

Also, all the other observed facts were found to confirm the idea of fluids, considered as heavy bodies exerting fluid pressure: thus, it was found—(7) that a fluid presses downwards on a lighter body which is entirely immersed; and presses upwards on a heavier body which is partially immersed; and presses in all

directions against surfaces, according to the deductive Propositions which we have demonstrated to obtain in a heavy fluid.

Proposition 12. *Inductive Principle* 2. Air has weight.

The facts included in this induction are such as the following:—

(1). We, existing in air, are not sensible of any weight belonging to it.

(2). Bubbles of air rise in water till they come to the surface.

(3). If we open a cavity, as in a pair of bellows, the air rushes in.

(5). If in such a case air cannot enter water can, the water is drawn in; as when we draw water into a tube by suction or into a pump by raising the piston.

(5). If a cavity be opened and nothing be allowed to enter, a strong pressure is exerted to crush the sides of the cavity together.

If facts (1) and (2) were explained at first by saying that the proper place of air is above water; that when it is in its proper place, as in (1), it does not gravitate (as in Proposition D.), but that when it is below its proper place, as in (2), it tends upwards to its place; the facts (3) (4) (5) were explained by saying that nature abhors a vacuum.

But it was found by experiment:

(6). That water could not by suction or by a pump be raised more than 34 feet; and stood at that height with a vacuum above it.

(7). That mercury was supported in a tube with a vacuum above it, at the height of 30 inches (Torricelli's experiment).

(8). That at the top of a high hill this column of mercury was less than 30 inches (Pascal's experiment).

These facts overturned the explanation derived from nature's horror of a vacuum; for men could not suppose that nature abhorred a vacuum less at the top of a hill than at the bottom, or less over 31 feet of water than over one foot.

But all the facts were distinctly explained and rigorously deduced by adopting the Idea of fluid pressure, and the *Principle* that air has weight, its weight producing its effects according to the laws of fluid pressure. This will be seen in the Deductive Propositions which we shall demonstrate as the consequences of assuming that air has weight.

The Inductive Proposition was further confirmed by—(9) experiments with the air-pump; for it appeared that as the receiver was exhausted the mercury in the Torricellian experiment fell.

Proposition 13. *Inductive Principle* 3. Air is elastic; and the elastic force of air at a given temperature varies as the density.

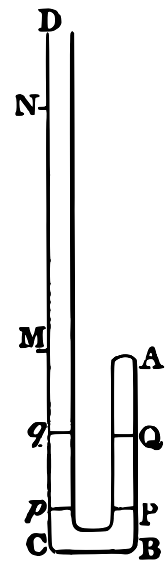
The facts which show air to be elastic are such as follow:—

(1). A bladder containing air may be contracted by pressure, and expands again when the pressure is removed.

(2). A tube closed above and open below, and containing air, being immersed in water, the air contracts as the immersion is deeper, and expands again when the tube is brought to the surface.

(3). If a close vessel, containing water and air, fitted with a tube making a communication between the water and the exterior, be placed in the exhausted receiver, the water is expelled through the tube.

The principle that the elastic force increases in proportion to the density, was experimentally proved (first by Boyle*) in the following manner:—



A uniform tube *ABCD* was taken, closed at *A* and open at *D*, and bent so that *BA* and *CD* were upright at the same time. Quicksilver was poured in, so that its ends stood at *M* and *P*, Again, more quicksilver was poured in, so that its ends stood at *N* and *Q*. And *Pp*, *Qq* being horizontal, *AQ* and *Nq* were measured.

And the observations of the results of this experiment were registered as in the two first columns of the annexed table,

(1) AQ in.	(2) Nq in.	(3) Barom. in.	(4) Press. in.
12	0	30	30
10	6	30	36
8	15	30	45
6	30	30	60
4	60	30	90

The whole pressure on the air *AQ* at *Q* is the pressure of the column of mercury *Nq*, together with the pressure of the atmosphere upon *N*. The latter pressure being taken to be equal to 30 inches of mercury, as in column (3), and added to *Nq*, we have, as in column (4), the pressure upon the air *AQ*. And it appears by comparing columns (1) and (4) that this pressure is always inversely as the space occupied by the air;

$$\text{for } 10 : 6 :: 60 : 36,$$

and so, for any other of the observations.

Now the pressure on the air *AQ* at *Q* is balanced by the elastic force of the air in *AQ*; these two forces acting upon the same surface, namely the surface of the mercury at *Q*. And the pressure upon *Q* has been found to be inversely as *AQ*; therefore, the elasticity of the air in *AQ* is inversely as *AQ*; that is, inversely as the space occupied.

The quantity of air remaining the same, the density is inversely as the space occupied: therefore, the elastic force is as the density.

Proposition 14. *Inductive Principle* 4. The elastic force of air is increased by an increase of temperature.

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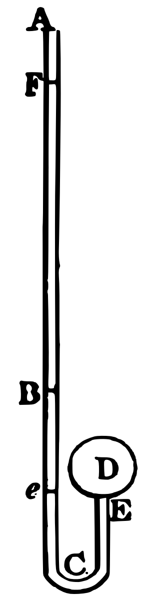
* Shaw's Byle, Vol. II. p. 671.

The facts included in this induction are such as the following:—

(1). If a bladder partly full of air be warmed it becomes more completely full.

(2). If an inverted vessel confining air in water be warmed the air escapes in bubbles.

It was experimentally ascertained *how much* the elastic force of air is increased by heat (first by Amontons*) in the following manner: —



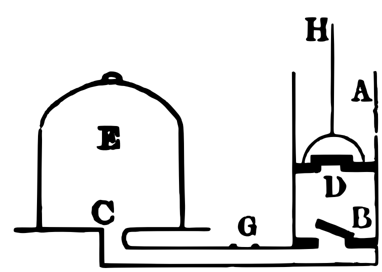
A bent tube *ABC*, with a bulb *D* containing common air, was filled with mercury from *B* to *E*, *B*, being 3 inches higher than the horizontal plane *Ec*, The bulb was then placed in boiling water, and it was found that a small portion of the mercury was driven out of the bulb, so that the extremity of the column was elevated to *F*, *BF* being 11 inches.

The air occupied very nearly the same space in the last case as in the first; for the bore of the tube was very small, and the surface of the mercury continued nearly in the same position at *E*. The pressure on the air in *D* at first is the pressure of the atmosphere (30 inches of mercury,) together with the weight of the column *Be* (3 inches;) therefore it is 33 inches. And the pressure on the air in *D* when immersed in boiling water is, in the same manner 30 inches, together with the weight of the column *Fe* (which is 14 inches;) that is, it is 44 inches; that is, air, in this experiment, has its elasticity increased from 33 to 44, by heating the water to boiling: that is, the elasticity was increased one third.

Proposition 15. To describe the construction of the air-pump and its operation.

A *valve* is an appendage to an orifice, closing it, and opening in such a manner as to allow fluid to pass through the orifice in one direction and not in the opposite direction.

A *piston* is a plug capable of sliding in an orifice or tube so as to produce or remove fluid pressure.



The *Air-pump* consists of a barrel and piston with valves, the *suction-pipe* communicating with a close vessel called the *receiver*.

Let *AB* be the barrel, *B* the inwards-opening valve at, the bottom of the barrel, *D* the piston with its outwards opening valve, *BC* the pipe, *E* the receiver.

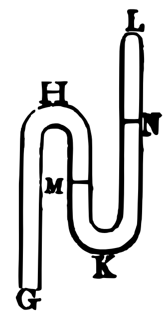
The piston *D* being in its lowest position, is raised to its highest position by the handle *H*. During the rise no air is admitted at *D*; and the air in *CD*, by its elasticity, expands and follows the piston in its ascent, passing through the valve *B*; and thus, air is drawn out of the receiver *E*.

The piston is then made to descend again to its lowest position: no air returns through the valve *B*, and the air in *BD* escapes by the valve at *D*.

* Mem. de l'Acad. Roy. des Sciences de Paris. 1699. p 113.

The piston is again raised, and more air is drawn out of *E* as before: and so on without limit.

Proposition E. To explain the construction of the siphon-gauge.

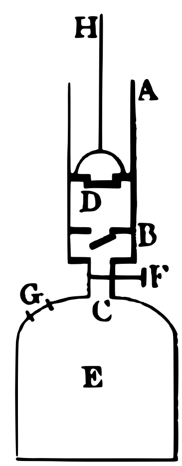


The *Siphon-gauge* is a twice-bent tube, closed at one end, and containing fluid, fixed to an air-pump *C* or other machine, to determine the degree of rarefaction of the air.

Let *GHKL*, closed at *L*, be the siphon-gauge, (fixed to *G* in the last or in the next Proposition), and let *MKN* be a portion of the tube filled with mercury, *LN* being a vacuum. Then the vertical height of *N* above *M* measures the density of the air in *GHM*.

If *GHM* were a vacuum (that is, if the exhaustion in the air-pump were complete) *M*, *N* would be at the same level.

Proposition 16. To describe the condenser and its operation.



The *Condenser* consists of a barrel and piston with valves, opening the contrary way from those of the air-pump, and communicating by a pipe with a closed receiver.

Let *AB* be the barrel, *B* the outwards-opening valve at the bottom of the barrel, *D* the piston with its inwards-opening valve, *BC* the pipe, *E* the receiver.

The piston *D* being in its highest position, is forced to its lowest position by the handle *H*. During the descent no air escapes through *D*, and the air in *BD* is driven through the valve *B*, and increases the quantity in the receiver *E*.

The piston is then made to ascend, and no air enters the barrel at *B*, because the valve opens outwards; but air enters the barrel *BD* by the valve *D*.

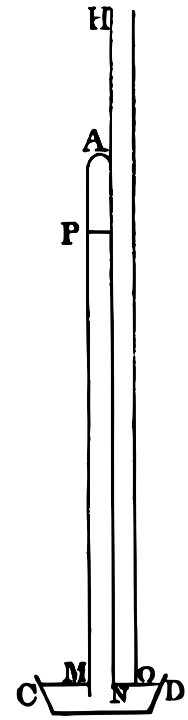
The piston is again forced down, and more air is driven into the receiver *E* as before: and so on without limit.

The pipe *BC* has a stop-cock *F*, and when this is closed, the pump may be screwed off, after the condensation is made.

Proposition 17. To explain the construction of the common barometer, and to show that the mercury in the tube is sustained by the pressure of the air on the surface of the mercury in the basin.

A *Barometer* is a (glass) tube, closed at one end and open at the other, which, being filled with a fluid (as mercury) is inverted with its open end in a basin. In any place, the fluid stands at a certain height (if the tube be long enough,) leaving a vacuum above.

Since the air has weight, it presses upon the surface *CD* of the mercury in the basin, and this pressure is resisted by the pressure of the column of mercury *PM*, arising from its weight. The mercury in the tube is sustained by the pressure of the mercury in the basin *CD*, which pressure again is sustained by the pressure of the atmosphere on the surface of the mercury in the basin.



Proposition 18. In the common barometer, the pressure of the atmosphere is measured by the height of the column of mercury above the surface of the mercury in the basin.

Let *AM* be the vertical tube, *A* its closed end, *CD* the basin and *MP* the height at which the fluid stands.

The upper parts of the atmosphere are less dense than the lower; but so long as the whole is in equilibrium, this condition does not affect the laws of fluid pressure; and Propositions 1, 2, 3, 4, 5, 6, of this Book will be still true.

Take, on the surface of the basin, *NO* equal to *NM*, the horizontal section of the tube; and suppose a tube *HN*, with vertical sides, standing on the base *NO*, to be continued upwards to the limits of the atmosphere. By Axiom 3, if all the rest of the atmosphere become rigid the pressure is not altered; and hence by Proposition 4., the pressure upon *NO* is equal to the weight of the column *HN*. But on this supposition, the pressures on *MN*, *NO* are equal, by Proposition 1. And the pressure on *MN* is equal to the weight of the vertical column of mercury *MP*. Therefore, the weight of the column of mercury is equal to the weight of the column of atmosphere on the same base. Therefore, the weight of the column of atmosphere is measured by the weight of the column of mercury; that is, the pressure of the atmosphere on a surface equal to the section of the tube made at the surface of the mercury in the basin, is equal to the weight of the vertical column of mercury which stands on the same section.

Therefore, the pressure of the atmosphere is measured by the weight of the column of mercury, that is, by the height, if the section and the density continue constant; for the weight of a column is as section × height × density.

Corollary: 1. If, instead of mercury, the tube be filled with any other fluid, as water, the fluid will stand at such a height as to support the weight of the atmosphere; and the height will be greater as the density of the fluid is less.

The mean height of the mercury-barometer being 30 inches, and the specific gravity of mercury 13.6, the mean height of the water-barometer is 13.6 × 30 inches 408 inches = 34 feet.

Corollary: 2. If the tube *AM* be not vertical, the Proposition is still true, the vertical height of *A* above *M* being still taken for the height of the fluid; for the pressure on *MN* is the same as if *AM* we vertical, by Proposition 2.

Corollary: 3. If the portion of the tube *AP*,* instead of being empty, contain air of less density than the atmosphere, a column of fluid *PM* will still be sustained, smaller than the column where *AP* is a vacuum; for if *P* were to descend to *M*, the pressure on *MN* would be less than the pressure on *NO*, which is impossible.

Corollary: 4. If a tube inclined or vertical, having its lower end, *M*† immersed in a fluid, and its upper end, *A*, closed, be full of water; the water will be supported if the vertical height of *A* above *M* be less than the height of the water-barometer.

In this case if *HM* be the height of the water-barometer, and if *AP*, drawn horizontal, meet *HM*, in *P*, the pressure upwards on the fluid at *M* would support a column of water of the vertical height *HM*. The pressure arising from the water in *AM* is equivalent to the weight of a column of the height *PM* (Proposition 4). Therefore, the pressure upwards at *M* will support the column *AM*; and will, besides, produce at *A*, a pressure upwards equal to the weight of a column *HP*.

Corollary: 5. If the interior of a tube *AM*, inclined or vertical; having its lower end M immersed in water, and its upper end A closed, become a vacuum, (that is, empty of air as well as other fluids) the fluid will rise in the tube; and will fill the whole tube or part of it, according as the vertical height of the closed end is less or greater than the height of the water-barometer.

For when the surface within the tube is not pressed by the column of fluid requisite to produce equilibrium, the pressure on the other parts of the surface will prevail, and drive the fluid into the vacuum. (Axiom 3.)

Proposition 19. To describe the construction of the common pump, and its operation.

The *Common Pump* consists of a cylindrical barrel *AB*, closed at bottom with an upwards-opening valve *B*, and of a piston *D* with an upwards-opening valve, which moves up and down in the barrel. A *suction-pipe BC* passes downwards from the valve *B* to the well at *C*, and the water which rises above the piston is delivered by the *spout E*.

The operation of the pump is as follows. The piston D being in its lowest position, is raised to its highest position by means of the lever *HKL*. Since the valve *D* opens upwards, no air is admitted at *D* during this rise; and since the valve *B* opens upwards, the air which occupied *CD* follows the piston in its ascent; in consequence of its elasticity (Proposition 13) it expands, and its pressure on the water at *C* is diminished. Hence the water in the suction-pipe rises by the pressure of the atmosphere on the surface of the well to some point *F*. (Corollary: 3 to Proposition 18).

The piston is then made to descend to its lowest position, the valve *B* is closed, and therefore the quantity of air in *FB* is not changed, and the water remains at *F*, while the air in *BD* escapes by the valve at *D*.

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* See fig. p. 114.

† See fig. p. 119.

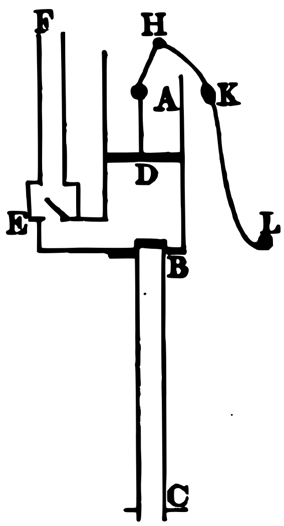
The piston is then again raised, the air in *DF* expands as before, and the surface of the water at *F* comes to a new position at *G*.

The same movements being repeated, the water will again rise; and so on, till it reaches the piston *D*, after which time the piston in its ascent will lift the water, and when it has lifted It high enough, will deliver it out at the spout *E*.

Corollary: The water in the common pump is raised by the weight of the atmosphere, and cannot be raised to a height greater than that of the water-barometer. (See Proposition 12.) The height of the water-barometer is 34 feet (Proposition 18. Corollary: 1.)

Proposition 20. To describe the construction of the forcing pump and its operation.

The *Forcing Pump* consists of a cylindrical barrel *AB*, closed at bottom with an upwards-opening valve *B*; of a piston *D* with no valve; and of a spout *E* with an outwards-opening valve. The piston moves up and down, and the suction-pipe descends from the bottom of the barrel to the well, as before, and the spout carries the water upwards.

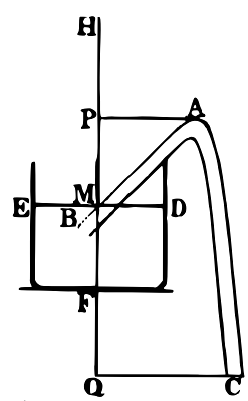


The operation of the pump is as follows. The piston *D*, in ascending from its lowest to its highest position, draws the water after it as in the common pump. When the piston descends, the air is forced out at the valve *E*; and after a certain number of ascents, the water comes into the barrel *AB*. When the piston next descends it forces the water through the valve *E*, and continues afterwards to draw the water through the valve *B* in its rise, and to extrude it through the valve *E* in its descent, by which means it is forced into the tube *EF*, which may be upright, or in any other position.

Corollary: By means of the forcing pump water may be raised to any height; the tube *EF* being prolonged upwards, and an adequate force applied to force the piston down wards.

Proposition 21. To describe the siphon and its action.

A *Siphon* is a bent tube, open at both ends, and capable of being placed with one end in a vessel of fluid, and the other end lower than the upper surface of the fluid in the vessel.



Let *BAC* be the bent tube placed so that the end *B* is immersed in the water *FED*, and the outer end *C* is below the surface *ED*.

If the tube *BAC* be filled with water, and if the vertical height of the portion *MA* be less than the height of the water-barometer, the tube will act as a siphon, that is, the water will constantly run through the tube *BAC* and out at *C*.

For the tube being filled with water, let the end *C* be stopped; and let *HM* be the height of the water-barometer; *AP*, *CQ*, horizontal. Suppose *HM* to be a column of water of the height of the water-barometer; and suppose the water in the tube *HMAC* to remain fluid, while all the rest becomes rigid: the pressures at *M*, *A*, and *C* will not be altered by this supposition (Axiom 5). But on this supposition the pressure downwards at *C* is equal to the height of a vertical column *HQ* (Proposition 2.) And the pressure of the atmosphere upwards at *C* is equal to the weight of a vertical column *HM* (Proposition 18. Corollary: 2). Therefore, the column *AC*, being acted upon by a pressure downwards equal to the weight of a column *HQ*, and a pressure upwards equal to the weight of a column *HM*, if the tube be opened at *C*, the former will prevail and the column *AC* will descend.

Also, the column *MA* will ascend, so that no interval shall exist in the fluid at A. For the interval, if any should take place, must be a vacuum, since the air has no access to it. And since the vertical height of *MA* is less than that of the water-barometer, by Corollary: 5 to Proposition 18., the fluid will rise in the tube and will fill this vacuum.

Therefore, the whole fluid *MAC* will move along the tube and flow out at *C*.

Corollary: 1. The fluid in the siphon *BAC* will be urged in the direction *BAC* by a force equal to a column of fluid *MQ*.

Corollary: 2. If the vertical height of *MA* be greater than that of the water-barometer, there will be a vacuum formed above the fluid at *A* (Proposition 18. Corollary: 5), and the siphon will not act.

Corollary: 3. Also, if instead of water and the water-barometer, we had taken any other fluid and the corresponding barometer, the reasoning, and the result, would have been the same as above.

Proposition F. *Inductive Principle* 5. Many (or all) fluids expand by heat and the amount of expansion at the heat at which water boils, and at the heat which ice melts, are each a fixed quantity.

The former part of this proposition is proved by including the fluids in bulbs, which open into a slender tube; for a small expansion of the fluid in the bulb is easily seen, when it takes place in the slender tube.

It was at first supposed, that when a fluid is exposed to heat, (as, for instance, when a vessel of water is placed on the fire,) a constant addition of heat takes place, increasing with the time during which the fire operates.

But it appeared, that when a tube containing air is placed in water thus exposed to heat, the expansion of the air (observed in the way described in Proposition 13) goes on till the fluid boils, after which no additional expansion takes place.

This fact is explained by assuming the expansion of air as the *Measure* of heat, and by adopting the Principle that the heat of boiling water is a fixed quantity.

This principle was first experimentally established by Amontons. Afterwards it was ascertained by Fahrenheit (1711), and others, that the expansion of oil, spirit of wine, mercury, at the heat at which water boils, is a fixed quantity; and hence Fahrenheit made the *boiling point* of water one of the fixed-points of his thermometers, which were filled with spirit of wine or with mercury.

For another fixed-point he took the cold produced by a mixture of ice, water, and salt; and he assumed this to be the *point of absolute cold*.

But it was found by Reaumur (1730), that the *freezing point* of water, or the melting point of ice, is more fixed than the point of absolute cold determined in the above manner. This was proved in the same manner in which the heat of boiling water had been proved to be a fixed-point. The *freezing point* was then adopted as one of the fixed-points of the measure of heat.

Proposition 22. To show how to graduate a common thermometer.

The common *Thermometer* is an instrument consisting of a bulb and a slender tube of uniform thickness, containing a fluid (as mercury or spirits of wine) which expands by heat and contracts by cold, so that its surface is always in the tube*.

Let the instrument be placed in boiling water, and let the point to which the surface of the fluid expands in the tube be marked as the boiling point.

Let the instrument be immersed in melting ice, and let the point to which the surface of the fluid contracts in the tube be marked as the *freezing point*.

For *Fahrenheit's division*, divide the interval between the freezing point and the boiling point into 180 equal parts; and continue the scale of equal parts upwards and downwards. Place 0 at 32 parts below the freezing point, 32 at the freezing point, 212 at the boiling point; and the other numbers of the series at other convenient points and the scale is graduated, the numbers expressing degrees of heat according to the place of the surface of the fluid in the tube.

For the *centigrade division*, divide the interval between the freezing and boiling points into 100 equal parts; mark the freezing point as 0 degrees, the boiling point as 100 degrees, and so on as before.

Proposition 23. To reduce the indications of Fahrenheit's thermometer to the centigrade scale, and the converse.

To reduce Fahrenheit to centigrade, subtract 32, which gives the number of degrees above the freezing point: and multiply by $\frac{5}{9}$, because 180 degrees of Fahrenheit are equal to 100 centigrade.

Thus

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* In practice, the part of the thermometer not occupied by the thermometrical fluid is rendered a vacuum.

$$59°F = 27°F \text{ above } 32°F = \frac{5 \times 27°}{9} \text{ centig. above } 0 = 15° \text{ cent.}$$

To reduce centigrade to Fahrenheit, multiply by $\frac{9}{5}$, which gives the number of Fahrenheit's degrees above the freezing point, and add 32, which gives the number above Fahrenheit's zero.

Thus

$$60° \ C. = 90° \ F \text{ above freezing} = 90° \ F + 32° \ F = 122° \ F.$$

Notes Respecting the Examinations

In the Preceding Subjects.

UNIVERSITY REGULATIONS.

Plan Of Examination for Questionists Who Are Not Candidates for Honors.

1. That the subjects of the Examination shall be the first fourteen, or the last fourteen Chapters of the Acts of the Apostles, and one of the longer, or two or more of the shorter Epistles of the New Testament, in the original Greek, one of the Greek and one of the Latin Classics, three of the six Books of Paley's Moral Philosophy, the History of the Christian Church from its Origin to the assembling of the Council of Nice, the History of the English Reformation, and such mathematical Subjects as are prescribed by the Grace of April 19, 1837, at present in force.

2. That in regard to these Subjects, the appointment of the Division of the Acts—of the Epistle or Epistles—of the Books of Paley's Moral Philosophy, and both of the Classical Authors and of the portions of their Works, which it may be expedient to select, shall be with the persons who appoint the Classical Subjects for the Previous Examination.

3. That public notice of the Subjects so selected for any year shall be issued in the last week of the Lent Term of the year next but One preceding.

4. That the Examination shall commence on the Wednesday preceding the first Monday in the Lent Term.

5. That on the Monday previous to the commencement of the Examination the Examiners shall publish the names of the persons to be examined, arranged in alphabetical order, and separated into two divisions.

6. That the distribution of the Subjects and Times of Ex-amination shall be according to the following Table:

	Div.	9 to 12.	Div.	12½ to 3½.
Wednesday	1	Euclid	2	Greek Subject
Thursday	1	Greek Subject	2	Euclid
Friday	1	Mechanics and Hydrostatics	2	Latin Subject
Saturday	1	Latin Subject	2	Mechanics and Hydrostatics
Monday	1	Paley and Eccles. History	2	Acts and Epistle or Epistles
Tuesday	1	Acts and Epistle or Epistles	2	Paley and Eccles. History
Wednesday	1	Arithmetic and Algebra	2	Arithmetic and Algebra

7. That the Examination shall be conducted entirely by printed papers.

8. That the Papers in the Classical Subjects and in the Acts and Epistles shall consist of passages to be translated, accompanied with such plain questions in Grammar, History, and Geography, as arise immediately out of those passages.

9. That the Papers in the Mathematical subjects shall consist of questions in Arithmetic and Algebra, and of Propositions in Euclid, Mechanics, and Hydrostatics, according to the annexed schedule.

10. That no person shall be approved by the Examiners, unless he show a competent knowledge of all the subjects of the Examination.

11. That there shall be three additional Examinations in every year; the first commencing on the Thursday preceding Ash-Wednesday, the second on the Thursday preceding the Division of the Easter Term, and the third on the Thursday preceding the Division of the Michaelmas Term.

12. That in these additional Examinations the distribution of the subjects and the hours of the Examination shall be at the discretion of the Examiners, the subjects being the same as at the Examination in the preceding January.

13. That no person shall be allowed to attend any Examination whose name is not sent by the Prælector of his College to the Examiners before the commencement of the Examination.

14. That in every year at the first Congregation after the 10th day of October, the Senate shall elect four Examiners, (who shall be Members of the Senate, and nominated by the several Colleges according to the cycle of Proctors and Taxors) to assist in conducting the Examinations of the three following terms.

15. That two of these Examiners shall confine themselves to the Classical Subjects, and two to Paley's Moral Philosophy, Ecclesiastical History, the Acts of the Apostles, and the Epistles.

16. That the two Examiners in the Mathematical Subjects, at the Examination in January, be as hitherto the Moderators of the year next but one preceding; and that at the other three Examinations the Moderators for the time being examine in the Mathematical Subjects.

17. That each of the six Examiners shall receive £20 from the University Chest.

18. That the Pro-Proctors and two at least of the Examiners attend in the Senate-House during each portion of the Examination in January.

19. That the first Examination, under the Regulations now proposed, [that is, in the theological subjects] shall take place in the Lent Term of 1846.

Schedule of Mathematical Subjects of Examination, for the degree of B.A of Persons not Candidates for Honors.

Arithmetic.

Addition, subtraction, multiplication, division, reduction, rule of three; the same rules in vulgar and decimal fractions: practice, simple and compound interest, discount, extraction of square and cube roots, duodecimals: together with the proofs of the rules and the reasons for the processes employed.[*]

Algebra.

1. Definitions and explanation of algebraical signs and terms.

2. Addition, subtraction, multiplication, and division of simple algebraical quantities and simple algebraical fractions.

[*] Added by Grace, March 20, 1846.

3. Algebraical definitions of ratio and proportion.

4. If $a : b :: c : d$ then $ad = be$, and the converse:

also, $b : a :: d : e$,

and $a : c :: b : d$

and $a + b : b :: c + d : d$.

5. If $a : b :: c : d$.

and $c : d :: e : f$,

then $a : b :: e : f$.

6. If $a : b :: c : d$,

and $b : e :: d : f$,

then $a : e :: c : f$.

7. Geometrical definition of proportion. (Euclid Book v. Definition 5.)

8. If quantities be proportional according to the algebraical definition, they are proportional according to the geometrical definition.

9. Definition of a quantity *varying* as another, *directly*, or *inversely*, or as two others *jointly*.

10. Easy equations of a degree not higher than the second involving one or two unknown quantities and questions producing such equations.*

Euclid's Elements.

Book 1. 2. 3.

Book 6. Propositions 1. 2. 3. 4. 5. 6.

Mechanics.

Definition of Force, Weight, Quantity of Matter, Density, Measure of Force.

The Lever.

Definition of the Lever.

Axioms.

Proposition 1. A horizontal prism or cylinder of uniform density will produce the same effect by its weight as if it were collected at its middle point.

Proposition 2. If two weights acting perpendicularly on a straight lever on opposite sides of the fulcrum balance each other, they are inversely as their distances from the fulcrum; and the pressure on the fulcrum is equal to their sum.

Proposition 3. If two forces acting perpendicularly on a straight lever in opposite directions and on the same side of the fulcrum balance each other, they are inversely as their distances from the fulcrum; and the pressure on the fulcrum is equal to the difference of the forces.

Proposition 4. To explain the kind of levers.

* Added by Grace, March 20, 1816.

Proposition 5. If two forces acting perpendicularly at the extremities of the arms of any lever balance each other, they are inversely as the arms.

Proposition 6. If two forces acting at any angles on the arms of any lever balance each other, they are inversely as the perpendiculars drawn from the fulcrum to the directions in which the forces act.

Proposition 7. If two weights balance each other on a straight lever when it is horizontal, they will balance each other in every position of the lever.

Composition and Resolution of Forces.

Definition of Component and Resultant Forces.

Proposition 8. If the adjacent sides of a parallelogram represent the component forces in direction and magnitude, the diagonal will represent the resultant force in direction and magnitude.

Proposition 9. If three forces, represented in magnitude and direction by the sides of a triangle, act on a point, they will keep it at rest.

Mechanical Powers.

Definition of Wheel and Axle.

Proposition 10. There is an equilibrium upon the wheel and axle when the power is to the weight as the radius of the axle to the radius of the wheel.

Definition of Pulley.

Proposition 11. In the single moveable pulley where the strings are parallel, there is an equilibrium when the power is to the weight as 1 to 2.

Proposition 12. In a system in which the same string passes round any number of pulleys and the parts of it between the pulleys are parallel, there is an equilibrium when power (P) : weight (W) :: 1 : the number of strings at the lower block.

Proposition 13. In a system in which each pulley hangs by a separate string and the strings are parallel, there is an equilibrium when P : W :: 1 : that power of 2 whose index is the number of moveable pulleys.

Proposition 14. The weight (W) being on an inclined plane and the force (P) acting parallel to the plane, there is an equilibrium when P : W :: the height of the plane : its length.

Definition of Velocity.

Proposition 15. Assuming that the arcs which subtend equal angles at the centers of two circles are as the radii of the circles, to show that if P and W balance each other on the wheel and axle, and the whole be put in motion, P : W :: W's velocity : P's velocity.

Proposition 16. To show that if P and P balance each other in the machines described in Propositions 11, 12, 13, and 14, and the whole be put in motion, $P : W :: W$'s velocity in the direction of gravity : P's velocity.

The Centre of Gravity.

Definition of Centre of Gravity.

Proposition 17. If a body balance itself on a line in all positions, the center of gravity is in that line.

Proposition 18 To find the center of gravity of two heavy points; and to show that the pressure at the center of gravity is equal to the sum of the weights in all positions.

Proposition 19. To find the center of gravity of any number of heavy points; and to show that the pressure at the center of gravity is equal to the sum of the weights in all positions.

Proposition 20. To find the center of gravity of a straight line.

Proposition 21 To find the center of gravity of a triangle.

Proposition 22 When a body is placed on a horizontal plane, it will stand or fall, according as the vertical line, drawn from its center of gravity, falls within or without its base.

Proposition 23. When a body is suspended from a point, it will rest with its center of gravity in the vertical line passing through the point of suspension.

Hydrostatics.

Definitions of Fluid; of elastic and non-elastic Fluids.

Proposition 1. Fluids press equally in all directions.

Proposition 2. The pressure upon any particle of a fluid of uniform density is proportional to its depth below the surface of the fluid.

Proposition 3. The surface of every fluid at rest is horizontal.

Proposition 4. If a vessel, the bottom of which is horizontal and the sides vertical, be filled with fluid, the pressure upon the bottom will be equal to the weight of the fluid.

Proposition 5. To explain the hydrostatic paradox.

Proposition 6. If a body floats on a fluid, it displaces as much of the fluid as is equal in weight to the weight of the body; and it presses downwards and is pressed upwards withe a force equal to the weight of the fluid displaced.

Specific Gravities.

Definition of Specific Gravity.

Proposition 7 If M be the magnitude of a body, S its specific gravity, and W its weight, $W = MS$.

Proposition 8. When a body of uniform density floats on a fluid, the part immersed : the whole body :: specific gravity of the body : the specific gravity of the fluid.

Proposition 9. When a body is immersed in a fluid, the weight lost : whole weight of the body :: the specific gravity of the fluid : the specific gravity of the body.

Proposition 10. To describe the hydrostatic balance, and to show how to find the specific gravity of a body by means of it, 1st, when its specific gravity is greater than that of the fluid in which it is weighed; 2nd, when it is less.

Proposition 11. To describe the common hydrometer, and to show how to compare the specific gravities of two fluids by means of it.

Elastic Fluids.

Proposition 12. Air has weight.

Proposition 13. The elastic force of air at a given temperature varies as to the density.

Proposition 14. The elastic force of air is increased by an increase of temperature.

Proposition 15. To describe the construction of the common air-pump, and its operation.

Proposition 16. To describe the construction of the condenser, and its operation.

Proposition 17. To explain the construction of the common barometer, and do show that the mercury is sustained in it by the pressure of the air on the surface of the mercury in the basin.

Proposition 18. The pressure of the atmosphere is accurately measured by the weight of the column of mercury in the barometer.

Proposition 19. To describe the construction of the common pump, and its operation.

Proposition 20. To describe the construction of the forcing-pump, and its operation.

Proposition 21. To explain the action of the siphon.

Proposition 22. To show how to graduate a common thermometer.

Proposition 23. Having given the number of degrees on Fahrenheit's thermometer, to find the corresponding number on the centigrade thermometer.

Also, such Questions and applications as arise directly out of the aforementioned Propositions of Mechanics and Hydrostatics*.

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* Added by Grace, March 20, 1846.

Examination Papers.

Mechanics and Hydrostatics.

Senate-house, Friday, Jan. 12, 1849. 9.-12.

First Division.

(A)

1. Define force, show how density is measured.

2. If two forces acting perpendicularly on a straight lever in opposite directions and on the same side of the fulcrum balance each other, they are inversely as their distances from the fulcrum; and the pressure on the fulcrum is equal to the difference of the forces.

3. One end of a given straight lever rests upon a fulcrum, and the other end is sustained by a force of 3 lbs. acting upwards, where must a weight of 12 lbs. be placed in order that there may be equilibrium?

4. Assuming that the resultant of two forces acting on a point lies along the diagonal of the parallelogram whose sides represent the forces in magnitude and direction, show that it is represented in magnitude by the diagonal.

Find the magnitude and direction of the resultant of two equal forces at right angles to one another.

5. In that system of pulleys in which each pulley hangs by a separate string, show that $P : W :: W$'s velocity : P's velocity.

6. When a body is placed on a horizontal plane, it will stand or fall, according as the vertical line, drawn from its center of gravity, falls within or without its base.

Construct a triangle upon a given horizontal base such that the vertical line through the center of gravity shall pass through an angle at the base.

7. Fluids press equally in all directions.

8. A given cubical vessel resting on one side in a horizontal position contains a given quantity of fluid, a body is placed in it which floats, the weight of the body being given, find the pressure on the base, and the height to which the fluid rises in the vessel.

9. When a body of uniform density floats on a fluid the part immersed : the whole body :: the specific gravity of the body : the specific gravity of the fluid.

10. If a cubic inch of iron weigh 4½ ounces, and a cubic foot of water 1000 ounces, what is the specific gravity of iron?

11. Explain the construction of the common barometer, and show that the mercury is sustained in it by the pressure of the air on the surface of the mercury in the basin.

12. If a barometer stands at 30 inches, what is the greatest vertical length of the suction-pipe of a common pump that will pump up mercury?

13. Show how a common thermometer is graduated.

“Temperate" is marked on Fahrenheit’s thermometer at 56°, what is its height on the centigrade?

MECHANICS AND HYDROSTATICS.

Senate-house, Friday, Jan. 12, 1849. 9.-12.

FIRST DIVISION.

(B)

1. Define weight, and show how a statical force is measured.

2. If two weights acting perpendicularly on a straight lever on opposite sides of the fulcrum balance each other, they are inversely as their distances from the fulcrum; and the pressure on the fulcrum is equal to their sum.

3. Two forces of 3 and 5lbs respectively act upon a given straight lever, where must the fulcrum be placed for equilibrium, supposing the forces to act in opposite directions?

4. If two forces acting on a point are represented in magnitude and direction by the two sides of a parallelogram, show that their resultant is represented in direction by the diagonal of the parallelogram.

If three forces acting on a point will keep it at rest, show that they will also when their insensity is doubled.

5 If P and W balance each other on the inclined plane, show that $P : W ::$ W’s velocity in direction of gravity : P’s velocity.

6. Find the center of gravity of two heavy points; supposing the two points rigidly connected and a fulcrum placed under the center of gravity, what is the pressure on the fulcrum? would the bodies be in equilibrium?

7. If a vessel, the bottom of which is horizontal and the sides vertical, be filled with fluid, the pressure upon the bottom will be equal to the weight of the fluid.

8 A crooked horn is filled with fluid, and a lid being placed over the top it is then inverted and made to rest upon its top; find the amount of pressure upon the lid

9 When a body is immersed in a fluid, the weight lost : whole weight of the body :: the specific gravity of the fluid : the specific gravity of the body.

10. A piece of wood which weighs 3lbs. and whose specific gravity : that of water :: 3 : 4 floats in water, what weight placed upon it would just sink it?

II. The pressure of the atmosphere is accurately measured by the weight of the column of mercury in the barometer

12 What difference will be produced in the action of a siphon by taking it to the top of a mountain where the barometer is 26 inches, the barometer below being at 30 inches?

13. Show how a centigrade thermometer is graduated.

A centigrade thermometer stands at 35°, what is the height of Fahrenheit’s?

NOTES.

MECHANICS AND HYDROSTATICS.

Senate-house, Friday, Jan. 12, 1849. 12½.-3½.

Second division.

(A)

I. If two weights acting perpendicularly on a straight lever on opposite sides of the fulcrum balance each other, they are inversely as their distances from the fulcrum; and the pressure on the fulcrum is equal to their sum.

The arms of a straight lever are 12 and 18 inches respectively; and a weight of 3lbs. is suspended at the extremity of the shorter arm, what is the pressure on the fulcrum?

2. If two weights balance each other on a straight lever when it is horizontal, they will balance each other in every position of the lever.

Is the converse 'necessarily' true? Why does the common balance not rest in all positions?

3. If three forces represented in magnitude and direction by the sides of a triangle act on a point, they will keep it at rest. If two forces of 5 lbs. and 12 lbs. act at right angles upon a point, find the magnitude of the force which will keep the point at rest. Find also the directions in which the two given forces must be applied, in order that the point may be kept at rest by the least possible force, and find its magnitude.

4. In a System of pulleys in which each pulley hangs by a separate string, and the strings are parallel, there is an equilibrium when $P : W :: 1$: that power of 2 whose index is the number of moveable pulleys.

5. If the weight W be supported on an inclined plane by a force P acting parallel to the plane, and the whole be put in motion, show that $P : W ::$ W's velocity in the direction of gravity : P's velocity.

6. Find the center of gravity of any number of heavy points, and show that the pressure on the center of gravity is equal to the sum of the weights in all positions.

Three weights 1 lb., 2 lbs., 3 lbs. are placed in a straight line at equal distances of 12 inches, find the distance of the common center of gravity from the middle weight.

7. When a body is suspended from a point, it will rest with its center of gravity in the vertical line passing through the point of suspension.

8. If a body floats in a fluid it displaces as much of the fluid as is equal in weight to the weight of the body; and it presses downwards and is pressed upwards with a force equal to the weight of the fluid displaced.

A prismatic solid whose height is five inches floats at a depth of three inches in a fluid, compare the specific gravities of the solid and fluid.

9. Describe the hydrostatic balance, and show how to find by means of it the specific gravity of a solid lighter than the fluid in which it is weighed.

10. The elastic force of air at a given temperature varies as the density.

11. Describe the construction of the condenser and its operation.

12. Explain the construction of the common barometer, and show that the mercury is sustained in it by the pressure of the air on the surface of the mercury in the basin.

What would be the effect 1^{st} of a hole at the bottom of the tube; 2^{nd} at the top?

13. Two vertical tubes are connected by a horizontal tube of 2 inches; supposing 12 inches of mercury poured into one tube, and 26 inches of water into the other; find the altitudes of the water and mercury in the two branches, the specific gravity of mercury being supposed 13 times the specific gravity of water.

NOTES.

MECHANICS AND HYDROSTATICS.

Senate-house, Friday, Jan. 12, 1849. 12½... 3½.

SECOND DIVISION.

(B)

1. If two forces acting perpendicularly on a straight lever in opposite directions and on the same side of the fulcrum balance each other, they are inversely as their distances from the fulcrum; and the pressure on the fulcrum is equal to the difference of the forces.

If the arms of the lever are 12 and 18 inches respectively, and a weight of 4 lbs. is suspended at the extremity of the longer arm, what is the magnitude and direction of the pressure on the fulcrum?

2. If two forces acting at any angles on the arms of any lever balance each other, they are inversely as the perpendiculars drawn from the fulcrum to the directions in which the forces act.

P and Q are two forces whose directions make equal angles with the arms of a bent lever; the lengths of the arms are 6 and 8 inches respectively; find the relation between P and Q when they balance each other.

3. If three equal forces act upon a point and keep it at rest, find the inclinations of their directions to each other. Find also the directions in which three forces represented by 3 lbs., 5 lbs., and 8 lbs., must be applied to a point so as to keep it in equilibrium.

4. In a system of pulleys in which the same string passes round any number of pulleys, and the parts of it between the pulleys are parallel, there is an equilibrium when the power : the weight :: 1 : the number of strings at the lower block.

5. Show that if P and W balance each other on the wheel and axle and the whole be put in motion, $P : W :: W$'s velocity : P's velocity.

6. If a body balance itself on a line in all positions, the center of gravity is in that line. If a body balance itself on a line in a certain position, what will be the position of the center of gravity?

7. Find the center of gravity of a triangle, and also of three equal weights placed at three angular points and show that they coincide.

8. The surface of every fluid at rest is horizontal.

9. Define a fluid, and prove that the pressure upon any particle of fluid of uniform density is proportional to its depth below the surface of the fluid. Two vessels are filled with fluid and placed upon a horizontal plane. The bases are 1 square foot and 2 square feet respectively, and altitudes 9 and 6 inches, compare the pressures upon the bases of the vessels.

10. When a body is immersed in a fluid, the weight lost : whole weight :: the specific gravity of the body : the specific gravity of the fluid.

If the specific gravities of iron and gold be 8 and 19 times the specific gravity of water respectively; find the weight in water of a substance combined of 1 lb. of iron and 1 lb. of gold.

11. A weight of 4 lbs. when placed upon a piece of wood whose specific gravity : that of water :: 3 : 5 just causes it to sink; find the weight of the wood.

12. Describe the construction of the common air-pump, and its operation.

13. The pressure of the atmosphere is accurately measured by the weight of the column of mercury in the barometer.

If 13 inches of water be inserted in the tube upon the mercury, what will be the altitude of the upper surface of the water when the common barometer stands at 30 inches, the specific gravity of mercury being supposed 13 times that of water? How much will the top of the water fall, when the mercurial-baromiter sinks an inch?

Appendix. Book 3. The Laws of Motion.

Remarks On Mathematical Reasoning

and the

Logic of Induction.

The Laws of Motion.

Definitions and Fundamental Principles.

1. The science which .treaty of Force producing Motion, and of the Laws of the Motion produced, is Dynamics.

2. In Dynamics, we adopt the Ideas, Definitions. Axioms, and Proposition of Statics.

3. We require also several new Ideas, Definitions, and Principles, which are obtained by Induction, and will be stated in the succeeding' Propositions.

4. Velocity is the degree in which a body moves quickly or slowly : thus, if a body describes a greater distance than another in the same time, it has a greater velocity.

5. The velocity of a body is uniform when it describes equal distances in all equal times.

6. The velocities of bodies, when uniform, are as the distances which they describe in equal times.

Definition 1. The velocity of a body moving uniformly is *measured* by the distances describe in a unit of time.

When the velocities of bodies are not uniform, they are increasing or decreasing.

Axiom 1. If a body move with an *increasing* velocity, the distance described in any time is *greater* than the distance which would have been described in the same time, if the velocity had continued uniform for the same time and the same as it was at the *beginning* of that time.

And the distance described in any time is less than the space which would have been described in the same time, if the velocity had been uniform for the same time and the same as it is at the *end* of that time.

Axiom 2. If a body move with a *decreasing* velocity the above Axiom is true, putting “less” for “greater” and “greater” for “less.”

Axiom 3. If two bodies move, having their velocities at every instant in a constant ratio, the distance described in any time by one body and by the other will be in the same ratio.

Axiom 4. If several detached material points, acted upon by any forces, move in parallel lines, parallel to the forces, in such a manner as to retain always the same distances from each other, and the same relative positions; they may be supposed to be rigidly connected, and acted upon by the same forces, and their motions will not be altered on this supposition.

Axiom 5. On the same supposition, the parallel forces may be supposed to be added together so as to become one force, and the motions will not be altered.

Axiom 6. When bodies in motion exert pressure upon each other, by means of strings, rods, or in any other way, the reaction is equal and opposite to the action at each point.

Definitions 2. (of Force), Definition 3 (of the Direction of Force), stand after Proposition 3; Definition 4 (of Uniform Force), stands after Proposition 3; Definition 5 (of Composition of Motions), after Proposition 8; Definition 6 (of Accelerating Force), after Proposition 13; Definition 7 (of Momentum), Definition 8 (of Elastic and Inelastic Bodies), Definition 9 (of Direct Impact) after Proposition 17 of this Book.

Axiom 7 stands after Proposition 2; Axiom 8 and 9 after Proposition 17 of this Book.

Proposition 1. In uniform motion the distance described with a velocity v in a time t is tv.

For (Definition 1.) v is the distance described in each unit of time, and t the number of units; therefore, the whole distance described is tv.

Proposition 2. *Inductive Principle* 1. First Law of Motion.

A body in motion, not acted upon by any force, will go on forever with a uniform velocity.

The facts which are included in this induction are such as the following:—

(1) All motions which we produce, as the motions of a body thrown along the ground, of a wheel revolving freely, go on for a certain time and then stop.

(2) Bodies falling downwards go on moving quicker and quicker as they fall farther.

It was attempted to explain these facts, by saying that motions such as (1) are *forced* motions, and motions such as (2) are *natural* motions and that forced motions decay and cease by their nature, while natural motions, by their nature, increase and become stronger.

But this explanation was found to be untenable; for it was seen—(3) that forced motions decayed less and less by diminishing the obvious obstacles. Thus a body thrown along the ground goes farther as we diminish the roughness of the surface; it goes farther and farther as the ground is smoother, and farther still on a sheet of ice. The wheel revolves longer as we diminish the roughness of the axis; and longer still, if we diminish the resistance of the air by putting the wheel in an exhausted receiver.

Thus, a decay of the motion in the cases (1) is constantly produced by the obstacles. Also, an increase of the motion in the cases (2) is constantly produced by the weight of the body.

Therefore, there is in these facts nothing to show that any motion decays or increases by its nature, independent of the action of external causes.

(3) By more exact experiments, and by further diminishing the obstacles, the decay of motion was found to be less and less; and there was in no case any remaining decay of motion which was not capable of being ascribed to the remaining obstacles.

Hence the facts are explained by introducing the *Idea of force*, as that which causes change in the motion of a body; and the *Principle*, that when a body is not acted upon by any force, it will move with a uniform velocity.

Corollary: 1. When a body moves freely (not being retained by any axis or any other restraint), and is not acted upon by any force, it will move in a straight line.

For since it is not acted on by any force, there is nothing to cause it to deviate from the straight line on any one side.

Definition 2. Force is that which causes change in the state of rest or motion of a body.

Definition 3. When a Force acts upon a body, and puts it in motion, the line of direction of the motion is the direction of the force.

Axiom 7. When a Force acts upon a body in motion, so that the direction of the force is the direction of the motion, the force will not alter the direction of the motion.

Proposition 3. *Inductive Principle* 2. Gravity is a uniform force.

The facts which are included in this induction are such as the following:—

(1). Bodies falling directly downwards fall quicker and quicker as they descend.

It was inferred, as we have seen in the last Proposition, that the additions of velocity in the falling bodies are caused by gravity.

An attempt was made to assign the law of the increase of velocity conjecturally, by introducing the Definition, that a uniform fore is a force which, acting in the direction of a body's motion, adds equal velocities in equal distances, and the Proposition that gravity is a uniform force.

The Definition is self-contradictory. But if it had not been so, the Proposition could only have been confirmed by experiment.

(2). It appeared by experiment that when bodies fall (down inclined planes) the distances described are as the squares of the times from the beginning of the motion.

This was distinctly explained and rigorously deduced by introducing the *Definition of uniform force*; that it is a force which, acting in the direction of the body's motion, adds equal velocities in equal times;

And the *Principle* that gravity (on inclined planes) is a uniform force.

For it may be proved deductively, as we shall see, that this definition being taken, the distances described in consequence of the action of a uniform force are as the squares of the times from the beginning of the motion. And if the force be other than uniform, the distances will not follow this law. Therefore, the Proposition, that gravity on inclined planes is a Uniform force, is the only one which will account for the results of experiment.

Also, if the force of gravity on inclined planes be a uniform force, the force of gravity when bodies fall freely is uniform, for when the inclined plane becomes vertical, he law must remain the same.

(3). The Proposition is further confirmed by shewing that its results, obtained deductively, agree with experiments made upon two bodies which draw each other over a fixed pulley (Atwood's Machine); and—.

(4) by the times of oscillation of pendulums.

Also, it appears that when gravity acts in a direction opposite to that of a body's motion, it subtracts equal velocities in equal times.

Hence, we introduce the following Definition.

Definition 4. A *uniform force* is that which, acting in the direction of the body's motion, adds (and in the opposite direction, subtracts,) equal velocities in equal times.

Proposition 4. If a uniform force act upon a body, moving it from rest, and if a be the velocity at the end of a unit of time, v, the velocity at the end of t units of time, is ta.

For the body will move in the direction of the force (Definition 3), and therefore the force is in the direction of the motion; and therefore, by Axiom 7, the direction of the motion is not altered by the action of the force. Hence by Definition 4, the velocity added to the velocity in each second is a, and in t seconds from the beginning of the motion it is ta.

Corollary: 1. At the end of $\frac{1}{n}$ of a unit of time the velocity is $\frac{a}{n}$.

Corollary: 2. At the end of $\frac{m}{n}$ units of time, the velocity is $\frac{ma}{n}$.

Corollary: 3. If v be the velocity at the end of the time t, the velocity at the end of the time $\frac{m}{n}t$ will be $\frac{m}{n}v$.

Proposition 5. If a uniform force act upon a body moving it from rest, and if a be the velocity at the end of a unit of time, d, the distances described at the end of t units of time, is ½ at^2.

Let each unit of time be divided into n equal portions; each of these will be $\frac{1}{n}$; and the whole number will be tn; and the velocity at the beginning of the first, second, third, fourth, etc. of these portions will be, by Proposition 1, Corollary: 2,

$$0, \frac{a}{n}, \frac{2a}{n}, \frac{3a}{n}, \text{ etc., } (tn \text{ terms}).$$

Suppose distances to be described in these portions of time with the velocity at the beginning of each portion continued uniform during that portion; these distances are by Proposition 1,

$$0 \times \frac{1}{n}, \frac{a}{n} \times \frac{1}{n}, \frac{2a}{n} \times \frac{1}{n}, \frac{3a}{n} \times \frac{1}{n} \ (tn \text{ terms})$$

which form an arithmetical series. And the last term of this series is

$$\frac{(tn-1)a}{n} \times \frac{1}{n}.$$

And the sum of it is (Introduction Article 60)

$$\frac{(tn-1)a}{n} \times \frac{1}{n} \times \frac{tn}{2};$$

$$\text{or } \frac{(tn-1)\ a}{n} \text{ or } \frac{at^2}{2} - \frac{at}{2n}.$$

In the same manner the velocity at the end of the first, second, third, etc. of these portions is

$$\frac{a}{n}, \frac{2a}{n}, \frac{3a}{n} \text{ etc. } (tn \text{ terms}).$$

Suppose distances to be described in these portions of time with the velocity at the end of each portion continued uniform during the time. These are as before

$$\frac{a}{n} \times \frac{1}{n}, \frac{2a}{n} \times \frac{1}{n}, \frac{3a}{n} \times \frac{1}{n}, \frac{4a}{n} \times \frac{1}{n} \ (tn \text{ terms});$$

an arithmetical progression, of which the last term is $\left(\frac{tna}{a} \times \frac{1}{n} + \frac{a}{n} \times \frac{1}{n}\right)\frac{tn}{2}$, or $\frac{(tn+1)\ at}{2n}$ or $\frac{at^2}{2} + \frac{at}{2n}$.

But in this case the body moves with a constantly increasing velocity. Therefore, by Axiom 1, the distance described in each of the times $\frac{1}{n}$ is greater than the distance described on the former of the above suppositions; and less than the distance described on the latter of the above suppositions. Hence the whole distance s is greater than $\frac{at^2}{2} - \frac{at}{2n}$, and less than $\frac{at^2}{2} + \frac{at}{2n}$.

Therefore, it is equal to $\frac{at^2}{2}$; for if not, let it be equal to a greater quantity, as $\frac{at^2}{2} + b$, and let $n = \frac{at}{2b}$: then $\frac{at}{2n} = b$; and therefore, the distance described is equal to $\frac{at^2}{2} + \frac{at}{2n}$; but it is less than this which is impossible. Therefore, the distance is not equal to a greater distance than $\frac{at^2}{2}$; and in like manner it may be shown that it is not equal to a less. Therefore, the distance is equal to $\frac{at^2}{2}$. Q. E. D.

Corollary: Hence if t, T, be any two times from the beginning of the motion and d, D the distances through which a body falls in those times, $d : D :: t^2 : T^2$.

Proposition 6 The distance described in any time from rest by the action of a uniform force is equal to half the distanced described by the last acquired velocity continued uniform for the time.

As in last Proposition, let t be the whole time, and a the velocity acquired in one unit of time. Then at is the velocity acquired in the whole-time t. And a body moving with this velocity for the time t would describe the distance at^2 by Proposition 1. But a body moving from rest by a uniform force describes the distance ½ at^2 by Proposition 5. Therefore, the latter distance is half the former, Q. E. D.

Corollary: 1. A body falling from rest by the uniform force of gravity, describes 16 feet in one second. Therefore, with the velocity acquired it would describe 32 feet in one second. Therefore, gravity generates a velocity of 32 feet in one second of time.

Corollary: 2. If g represent 32 feet, the distance through which a body falls in t seconds by the action of gravity is ½ gt^2.

Proposition 7. When a body is projected in a direction opposite to the direction of a uniform force, with a velocity v, the whole time (t) of its motion till its velocity cancels out, and the distance (d) described in that time, are known by the equations $v = at$, $d = ½\, at^2$.

For by the Definition of uniform force, the force, acting in a direction opposite to the motion, subtracts in equal times the same velocities which the same force adds when it acts in the direction of the motion. Therefore, at a series of units of time the velocities will be v, $v - a$, $v - 2\,a$, $v - ta$, till $v - ta$ becomes 0, when all the velocity is canceled out; and when this occurs, $v = ta = 0$, or $v = ta$.

Also, by Axiom 2, the same reasoning would hold as in Proposition 5, putting less for greater and greater for less; and therefore, the same conclusion is true, namely, $d = at^2$.

Proposition 8. *Inductive Principle* 3. Second Law of Motion. When any force acts upon a body in motion, the motion which the force would produce in the body at rest is compounded with the previous motion of the body.

The facts which the Induction includes are, in the first place, such as the following

(1). A stone dropped by a person in motion, is soon left behind him. .;

From (1) it was inferred that if the earth were in motion, bodies dropt or thrown would be left behind.

But if appeared that the stone was not left behind while it was moving in free space, and was only stopt when it came to the ground. Again, it was found by experiment,

(2). That a stone dropt by a person in motion describes such a path that relatively to him, it falls vertically.

(3). A man throwing objects and catching them again uses the same effort whether he be at rest or in motion.

Again, such facts as the following were considered:

(1). A stone thrown horizontally or obliquely describes a bent path and comes to the ground.

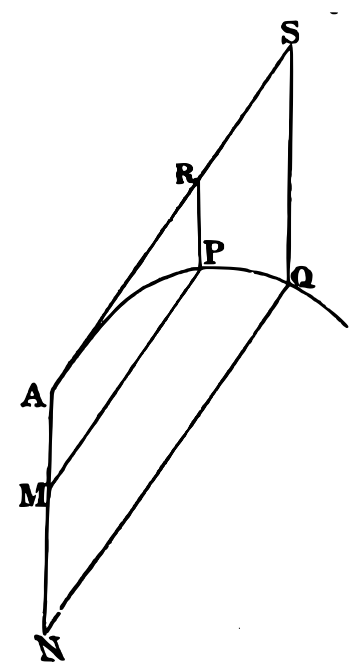
It was at first supposed that the stone does not fall to the ground till the original velocity is expended. But when the First Law of Motion was understood, it was seen that the gravity of the stone must, from the first, produce a change in the motion, and deflect the stone from the line in which it was thrown. And by more exact examination it appeared that (making allowance for the resistance of the air),—(5) the stone falls below the line of projection by exactly, the distance through which gravity in the same time would draw it from rest.

These facts were distinctly explained and rigorously deduced by introducing the *Definition of Composition of Motions*;—that two motions are compounded when each produces its separate effect in a direction parallel to itself;

And the Principle, that when a force acts upon a body in motion, the motion which the force would produce in the body at rest is compounded with the-previous motion of the body.

The Proposition is confirmed by shewing that its results, deduced by demonstration, agree with the facts.

Definition 5. Two motions are compounded when each produces its separate effect in a direction parallel to itself.



Proposition 9. If a body be projected in any direction and acted upon by gravity, in any time it will describe a curve line of which, the tangent intercepted by the vertical line, and the vertical distance from the tangent, are respectively the distances due to the original velocity and to the action of gravity in that time.

Let *AR* be the direction of projection; and in any time, let *AR* be the distance which the body would have described with the velocity of projection in that time, and *AM* the distance through which the body would have fallen in the same time. Then, completing the parallelogram *AMPR*, the body will, by the Second Law of Motion (Proposition 8) be found at *P*, and *RP* is equal to *AM*. Also *AR* is a tangent to the curve at *A*, because at *A* the body is moving in the direction *AR*. Therefore, etc. Q. E. D.

Corollary: If *P*, *Q* be the points at which the projectile is found, at any two times t, T from its being at *A*, and if *PR*, *QS* be vertical lines, meeting the tangent at *A* in *P*, then

$$PR : QS :: AR^2 : AS^2.$$

For $PR : QS :: t^2 : T^2$ by Corollary: to Proposition 5.

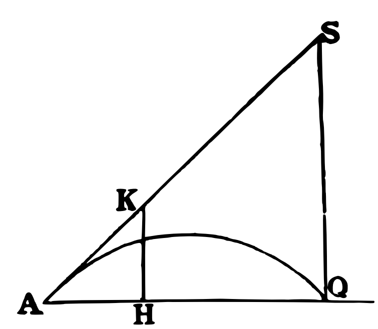
But $t : T :: AR : AS$; whence

$$t^2 : T^2 :: AR^2 : AS^2.$$

Therefore $PR : QS :: AR^2 : AS^2$.

Proposition 10. A body is projected from a given point in a given direction; to find the range upon a horizontal plane, and the time of flight.

The *range* is the distance from the point of projection to the point where the *projectile* (or body projected) again strikes a plane passing through the point of projection.



Let a body be projected in a direction *AK*, such that, *AH*, *HK* being horizontal and vertical, $AH : HK :: m : n$. Hence

$$\frac{AH}{HK} = \frac{m}{n}, \frac{AK^2}{HK^2} = 1 + \frac{AH^2}{HK^2} = 1 + \frac{m^2}{n^2} = \frac{n^2 + m^2}{n^2};$$

$$\frac{HK}{AK} = \frac{n}{\sqrt{n^2 + m^2}}; \frac{AH}{AK} = \frac{m\,HK}{n\,AK} = \frac{m}{\sqrt{n^2 + m^2}}.$$

Let v be the velocity of projection, *AQ* the path described, QS vertical; and let the time of describing *AQ* be t. Therefore, by the last Proposition, *AS*, the distance described with velocity v in the time t, will be vt. Also *SQ*, the distance fallen by gravity in the time t, will be $\frac{1}{2}\,gt^2$, by Proposition 6, Corollary: 2.

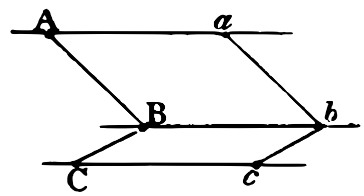
And $\frac{SQ}{AS} = \frac{HK}{AK}$; that is, $\frac{\frac{1}{2}gt^2}{vt} = \frac{n}{\sqrt{n^2 + m^2}}$

$\frac{gt}{2v} = \frac{n}{\sqrt{n^2 + m^2}}$, $t = \frac{2v}{g} \times \frac{n}{\sqrt{m^2 + n^2}}$; which is the time of flight,

Also $\frac{AQ}{AS} = \frac{AH}{AK}$, or $\frac{AQ}{vt} = \frac{m}{\sqrt{m^2 + n^2}}$, $AQ = vt \times \frac{m}{\sqrt{m^2 + n^2}}$; $AQ = \frac{2v^2}{g} \times \frac{mn}{m^2 + n^2}$, which is the range.

Proposition 11. If any particles, moving in parallel directions, and acted upon each by a certain force in the direction of its motion, move with velocities which are equal for all the particles at every instant, the motions of the particles will be the same if we suppose them to be connected so as to form a single rigid body, and the forces to be added together so as to form a single force.

Let *A*, *B*, *C*, be any particles acted upon by any forces, and moving in parallel directions with velocities which are equal at every instant. Since the velocities at every instant are equal, the distances described in the same time are equal for all the particles, by Axiom 3.



Let *Aa*, *Bb*, *Cc* be the distances described in any time, which are therefore equal and parallel. Therefore, *ab* is equal and parallel to *AB*, and *bc* to *BC*, and so on. Therefore, the relative positions and distances of the particles *A*, *B*, *C* are not altered by their motion into the places *a*, *b*, *c*.

Therefore, by Axiom 4, if we suppose the particles *A*, *B*, *C* to be rigidly connected, their motions will not be altered that is, the motions will not be altered if *A*, *B*, *C* are supposed to be particles of a single rigid body.

Also, by Axiom 5, if we suppose the forces which act upon the particles, *A*, *B*, *C*, to be added together so as to form a single force, the motion will not be altered.

Therefore, etc. Q. E. D.

Proposition 12. If, on two bodies, two pressures act, which are proportional to the quantities of matter in the two bodies, the velocities produced, in equal times in the two bodies are equal.

Let *P*, *Q*, be two pressures, and *m*, *n* two bodies, measured by the number of units of quantity of matter which each contains; and let $P : Q :: m : n$.

Let the pressure P be divided into m parts, each of which will be $\frac{P}{m}$, and let each of these parts of the force act upon a separate one of the *m* units into which the body m can be divided, and let it produce in a time *t* a velocity *v*. Each of the pressures $\frac{P}{m}$ will produce in the unit upon which it acts for the time an equal velocity *v*, in a direction parallel to *P*. Therefore, if all the *m* pressures act for the same time *t* upon the *m* units of the body respectively, all the units will move with velocities which are equal at every instant. Therefore, by Proposition 11, if we

suppose the m units to be connected so as to form one rigid body m, and the forces to be added so as to form a single force P, the motion will still be the same. That is, the pressure P acting upon the body m, will produce the velocity v in the time t.

In the same manner it may be shown that the pressure Q acting upon the body n will produce the same velocity which a pressure $\frac{Q}{n}$ produces in a body 1. But since $P : Q :: m : n$, $\frac{Q}{n} = \frac{P}{m}$; therefore $\frac{Q}{n}$ acting upon a body 1 will produce a velocity v in a time t. Therefore, Q acting on n will produce a velocity v in a time t; the same which P produces in m. Q. E. D.

Proposition 13. *Inductive Principle* 4. The Third Law of Motion. When pressure generates (or cancels out) motion in a given body, the accelerating force is as the pressure.

The facts included in this Induction are such as the following:—

(1). When pressure produces motion, the velocity produced is greater when the pressure is greater.

In order to determine in what proportion, the velocity increases with the pressure, further consideration and inquiry are necessary.

It appeared that,

(2). On an inclined plane the velocity acquired by falling down the plane is the same as that acquired by falling freely down the vertical height of the plane (Galileo's experiment).

(3). When two bodies P, Q* hang over a fixed pulley, the heavier P descends, and the velocity generated in a given time is as $P - Q$ directly, and as $P + Q$ inversely (Atwood's Machine).

(4). The small oscillations of pendulums are performed in times which are as the square roots of the lengths of the pendulums.

(5). In the impacts of bodies, the momentum gained by the one body is equal to the momentum lost by the other (Newton's Experiments).

(6). In the mutual attractions of bodies the center of gravity remains at rest.

These results are distinctly Explained and rigorously deduced by introducing the *Definition* of uniform Accelerating Force;—that it is as the velocity generated (or canceled out) in a given time;

And the *Principle* that the Accelerating Force for a given body is as the pressure.

Most of these consequences will be proved in the succeeding Propositions, (14, 15, 16, 17, 18), and thus this Inductive Proposition is confirmed.

Definition 6. Uniform Accelerating Force is *measured* by the velocity generated in a unit of matter in a unit of time.

Hence in the formula in Proposition 4 and 5, a represents the Accelerating Force.

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* See figure to Prop. 17.

Axiom 8. If two bodies move so that their velocities at every instant are equal, the Accelerating Forces of the two bodies at every instant are equal; and conversely.

Axiom 9. If two bodies move so that their Accelerating Forces at every instant are in a constant ratio, and are in the direction of the motion, the velocities added or subtracted in any time are in the ratio of the Accelerating Forces.

Proposition 14. In different bodies, the Accelerating Force is as the pressure which produces motion directly, and as the quantity of matter moved inversely.

Let two pressures P, Q, produce motion in two bodies of which the quantities of matter are M, N. Let $M : N :: P : X$; therefore, by Proposition 12, the force X would, in a given time, produce in N the same velocity which P would produce in M; that is, the Accelerating Force on M arising from the pressure P, is equal to the Accelerating Force on N arising from the pressure X.

But by the Third Law of Motion (Proposition 13) the Accelerating Force on N arising from the pressure X is to the Accelerating Force on the same body N arising from the pressure Q as X is to Q.

Therefore, the Accelerating Force on M arising from P is to the Accelerating Force on N arising from Q as X is to Q.

But $M : N :: P : X$; therefore $X = \frac{PN}{M}$, and therefore, X is to Q, as $\frac{PN}{M}$ is to Q, or as $\frac{P}{M}$ to $\frac{Q}{N}$.

Therefore, the Accelerating Forces of P on M and of Q on N are as $\frac{P}{M}$ and $\frac{Q}{N}$. Q. E. D.

Proposition 15. On the inclined plane, the time of falling down the plane is to the time of falling freely down the vertical height of the plane as the length of the plane to its height.

Let L be the length of the plane, H its height. The pressure which urges a body down an inclined plane is equal to the pressure which would support it acting in the opposite direction; but this pressure : W, the weight of the body :: $H : L$ (Book 1. Proposition 20.) Therefore, the pressure which produces motion on the plane is $\frac{WH}{L}$.

The quantity of matter of the body is as W.

Hence, since by the last Proposition the Accelerating Force on the inclined plane is as the pressure directly and the quantity of matter inversely; therefore Accelerating Force on Inclined Plane : Accelerating Force of body falling freely :: $\frac{WH}{WL} : \frac{W}{W} :: H : L$.

Now the force on the inclined plane is a uniform accelerating force; and therefore, the velocity acquired in a unit of time measures it, by Definition 6. Therefore, if La be the velocity acquired in a unit of time by a body falling freely, Ha will be the velocity acquired in a unit of time down the. inclined plane. And the

rule of Proposition 5 is applicable. Therefore, if t and t' be the times of falling down L, and of falling vertically down H,

$$L : H :: \tfrac{1}{2} H a t^2 : \tfrac{1}{2} L a t'^2;$$

$$\text{or } L^2 : H^2 :: t^2 : t'^2;$$

$$\text{or } L : H :: t : t'. \; .$$

Proposition 16. On the inclined plane, the velocity acquired by falling down the inclined plane is equal to the velocity acquired by falling freely down the vertical height of the plane.

As before, the accelerating force on the plane is to the accelerating force of a body falling freely

$$:: H : L;$$

Also, $s = \tfrac{1}{2} vt$, by Proposition 6; whence as before, v' being the velocity acquired by falling freely down H,

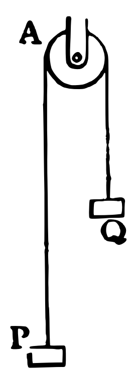
$$L : H :: \tfrac{1}{2} vt : \tfrac{1}{2} v' t' :: vt : v't';$$

But H : L :: $t' : t$ by Proposition 15.

Therefore $1 : 1 :: v : v'$;

Whence $v = v'$; the velocity acquired down the plane is equal to the velocity acquired down the vertical height, Q. E. D.

Proposition 17. When two bodies P, Q hang over a fixed pulley, and move by their own weight merely*, the heavier P descends, and the lighter Q ascends, by the action of an accelerating force which is as $\frac{P-Q}{P+Q}$.



The string which connects P and Q exerts an equal pressure in opposite directions upon P and upon Q, (Axiom 6). Let this pressure be X. Then since P is urged downwards by a force P and upwards by a force it is on the whole urged downwards by a force $P - X$. And the quantity of matter is P. Therefore, by Proposition 14, the Accelerating Force upon P downwards is as $\frac{P-X}{P}$. In the same manner, since X acts upwards upon Q and the weight of Q acts downwards, the accelerating force upon Q upwards is as $\frac{Q-X}{Q}$. But the accelerating force upon Q upwards and upon P downwards must be equal, because they move at every point with equal velocities, by Axiom 8.

Therefore $\frac{X-Q}{Q}$ is equal to $\frac{P-X}{P}$;

that is, $\frac{X}{Q} - 1$ is equal to $1 - \frac{X}{P}$;

* That is, neglecting the effect of the matter in the pulley and the string.

or $\frac{X}{Q} + \frac{X}{P}$ is equal to 2.

Therefore $\frac{A(P + Q)}{PQ}$ is equal to 2;

and X is equal to $\frac{2\ PQ}{P}$.

Hence $P - X$ is $P - \frac{2\ PQ}{P + Q}$, or $\frac{P^2 - PQ}{P + Q}$; and the accelerating force upon P, which is as $\frac{P - X}{P}$, is as $\frac{P - Q}{P + Q}$.

And, in like manner, the accelerating force upon Q is as $\frac{P - Q}{P + Q}$.

Definition 7. The *momentum* of a body is the product of the numbers which express its velocity and its quantity of matter.

Definition 8. Elastic bodies are those which separate-when one impinges upon another; inelastic bodies are those which do not separate after impact.

Definition 9. The impact of two bodies is direct, when the bodies, before impact, either moving in the same direction or one of them being at rest, the pressure which each exerts upon the other is in the direction of the motion.

Proposition 18. In the direct impact of two bodies the momenta gained and lost are equal.

Let P impinge upon Q directly, and let X be the pressure which each exerts upon the other at any instant. Therefore, the accelerating forces which act upon P and Q in opposite directions are as $\frac{X}{P}$ and $\frac{X}{Q}$: and are therefore at every instant in the constant ratio of $\frac{1}{P}$ to $\frac{1}{Q}$, or of Q to P. Therefore, by Axiom 9, the velocities generated in Q and canceled in P, in any time, are in the same constant ratio of Q to P. And the quantities of matter are as P and Q. Therefore, by Definition 7, the momentum generated in Q and the momentum canceled in P, in any time, are as PQ to PQ; that is, they are equal, Q. E. D.

Corollary: 1. If P and Q are elastic, they will separate after the impact; and the momenta generated and canceled in Q and P by the elasticity will still be equal, for the same reasons as before.

Corollary: 2. The velocity canceled in P, according to Corollary: 1, may be greater than its whole velocity. In this case, P will, after the impact, move in the opposite direction with a velocity which is the excess of the velocity lost over the original velocity.

Corollary: 3. Before the impact, Q may move in a direction opposite to P. In this case the velocity gained by Q is to be understood as including the velocity in the opposite direction, which is canceled.

Corollary: 4. If two bodies P and Q, move in opposite directions with velocities which are in the ratio of Q to P, they will be at rest after impact if they are inelastic, For since they are inelastic, they will not separate after impact: therefore they will either be at rest or move on together. But if they move in the direction of P's motion, P has lost less than its whole velocity, and Q has gained more than its own velocity. But this is impossible, for the velocities lost and gained are in the ratio of Q to P; that is, in the ratio of P's velocity to Q's velocity. Therefore, the bodies do not move in the direction of P's motion. And, in like manner, it may be shown that they do not move in the direction of Q's motion. Therefore, they remain at rest.

Proposition 19. The mutual pressure, attraction, or repulsion, or direct impact of two bodies, cannot put in motion their center of gravity.

Lot two bodies P, Q, act upon each other by pressure, attraction, or repulsion, the force which each exerts upon the other (Axiom 6) being X. Therefore (Proposition 14) the accelerating forces which, act on P and Q are as $\frac{X}{P}$ and $\frac{X}{Q}$ respectively, or in the constant ratio of Q to P. Therefore, the velocities acquired by P and Q in any equal times are in this ratio by Axiom 9, and therefore the distances are in the same ratio by Axiom 3.

Let P, Q, be any two bodies of which the center of gravity is C, which is p at first at rest. Therefore, by Book 1, Proposition 24, $Q : P :: CP : CQ$, and $CQ = \frac{P}{Q} CP$. And if Pp, Qq be any distances described in equal times, by the mutual

pressure, attraction, or repulsion of the bodies, it has been proved that $Q : P :: Pp : Qq$; and therefore $Qq = \frac{P}{Q} Pp$. Hence, subtracting, it follows that $Cq = \frac{P}{Q} Cp$, or $Q : P :: Cp : Cq$. And therefore, C is still the center of gravity of the bodies P, Q, when they are come into the positions p, q; that is, the center of gravity has not been put in motion.

Also, if the two bodies P, Q, not attracting or repelling each other, move towards each other with uniform velocities which are in the ratio of Q to P, and impinge; the distance described in any time (as Pp, Qq) will be in the same ratio of Q to P, and, as above, the center of gravity will be at rest. And when the bodies impinge on each other, the velocities of each will either be canceled, or canceled and generated in an opposite direction; and in either case, since the mutual pressure is equal on both, the accelerating forces which cancel and generate velocity, will be in the ratio of Q to P, as in Proposition 17. Therefore, the velocities canceled and generated are in the same ratio as the original velocities. Therefore, if the whole velocity of one body is canceled, the whole velocity of the other body also is canceled, and the bodies are both at rest, and their center of gravity is still at rest after impact.

But if the velocities be canceled, and velocities generated in an opposite direction, these new velocities will also be in the ratio of the original velocities, because the accelerating forces at every instant are so, (Axiom 9); and therefore,

the distances described in any time by the new velocities will be in the same ratio; and therefore, as before, it may be shown that C is still the center of gravity of P, Q.

Therefore, under all the circumstances stated, the center of gravity remains at rest, Q. E. D.

Examples to Propositions 4, 5, 6, 7; 10, 17, 18.

By means of these Propositions, we can solve such Examples as the following:—

When body falls freely by the action of gravity, the quantity a in Proposition 4 is 32 feet, the unit; of time being one second, and $v = gt$, also (Proposition 6, Corollary: 2) $v = ½\ gt^2$.

Example. 1. To find the velocity acquired by a body which falls by gravity for 30 seconds.

$$v = gt = 32 \times 30 = 960 \text{ feet per second.}$$

2. To find the distance fallen through in the same tune, $s = ½\ gt^2 = 16 \times 30^2 = 14400$ feet.

3. To find in what time a body falls through 1024 feet.

$$1024 = 16 \times t^2,\ t^2 = 64,\ t = 8 \text{ seconds.}$$

4. To find the velocity acquired in the same distance, $v = gt = 32 \times 8 = 256$ feet per second.

5. A body is projected directly upwards, with a velocity of l000 feet a second; how high with it go?

By Proposition 7, the height will be that through which a body must fall to acquire the same velocity.

Now since

$$v = gt,\ 1000 = 32t,\ t = \frac{1000}{32} = \frac{125}{4}\ 31¼''.$$

$$s = ½\ gt^2 = \frac{16\ (125)^2}{4^2} = (125)^2 = 15625 \text{ feet.}$$

6. A body is projected with a velocity of 32 feet a second in a direction which makes with the horizon half a right angle: to find the flight and the range.

In this case $m = n$; therefore, by Proposition 10,

$$\frac{n}{\sqrt{n^2 + m^2}} = \frac{1}{\sqrt{2}};\ t = \frac{2v}{g} \times \frac{1}{\sqrt{2}} = \frac{2 \times 32}{32 \times \sqrt{2}}\ t = \frac{2 \times 1600}{32} \times$$

$$= \sqrt{2} = 1.4 \text{ seconds;}$$

$$\text{the range} = \frac{2v^2}{g} \times \frac{mn}{m^2 + n^2} = \frac{2 \times (32)^2}{32} \times \frac{1}{2} = 32 \text{ feet.}$$

7. A cannon ball is projected with a velocity of 1600 feet a second, in a direction which rises 3 feet in 4 feet horizontal: find the time of flight and the range.

$$\frac{n}{\sqrt{n^2 + m^2}} - \frac{3}{\sqrt{9 + 16}} = \frac{3}{5};\ t = \frac{2 \times 1600}{32} \times \frac{3}{5} = 75 \text{ seconds}$$

the range $= \frac{2v^2}{g} \times \frac{nm}{m^2 + n^2} = \frac{2(1600)^2}{32} \times \frac{12}{25} = 768OO$ feet.

8. An inelastic body A impinges directly on another inelastic body B at rest, with a velocity of 10 feet a second; A being 3 and B 2 ounces, find the velocity after impact.

If x be the velocity of both after impact, the velocity lost by A is $10 - x$, and the velocity gained by B is x. Hence the momentum lost by A is $3 \times (10 - x)$; and that gained by B is $2 \times x$; and these are equal by Proposition 18; therefore

$$3(10 - \times) = 2x,\ 30 = 3x + x = 6.$$

9. The bodies being perfectly elastic, find the motions after impact.

In perfectly elastic bodies, the velocity lost by A and the velocity gained by B in the restitution of the figure are equal to the velocity lost by A and gained by B in the compression.

Now the velocity lost by A In the compression is 10 − 6 or 4; therefore the whole velocity lost by A is 8, and its remaining velocity 2.

And the velocity gained by B in the compression is 6, and therefore the whole velocity gained by B is 12, which is B's velocity after impact.

10. A body A (3 ounces) draws B (2 ounces) over a fixed pulley: find the distance described in one second from rest.

By Proposition 17, the accelerating force is as $\frac{3 - 2}{3 + 2}$; that is, it is ⅕ of gravity; and the distance in a second is as the force: therefore, the distance described in one second is $\frac{16}{5}$, or 3⅕ feet.

Remarks On Mathematical Reasoning,

and on

The Logic of Induction.

Section 1. *On the Grounds of Mathematical Reasoning.*

1. The study of a science, treated according to a rigorous system of mathematical reasoning, is useful, not only on account the positive knowledge which may be acquired on the subjects which belong to the science, but also on account of the collateral effects and general bearings of such a study as a discipline of the mind and an illustration of philosophical principles.

Considering the study of the mathematical sciences with reference to these latter objects, we may note two ways in which it may promote them;—by habituating the mind to strict reasoning,—and by affording an occasion of contemplating some of the most important mental processes and some of the most

distinct forms of truth. Thus, mathematical studies may be useful in teaching practical logic and theoretical metaphysics. We shall make a few remarks on each of these topics.

2. The study of Mathematics teaches strict reasoning—by bringing under the student's notice prominent and clear examples of trains of demonstration;— by exercising him in the habits of attentive and connected thought which are requisite in order to follow these trains;—and by familiarizing him with the peculiar and distinctive conviction which demonstration produces, and with the rigorous exclusion of all considerations which do not enter into the demonstration.

3. Logic is a system of doctrine which lays down rules for determining in what cases pretended reasonings are and are not demonstrative. And accordingly, the teaching of strict reasoning by means of the study of logic is often recommended and practiced. But in order to show the superiority of the study of mathematics for this purpose, we may consider,—that reasoning, as a practical process, must be learnt by practice, in the same manner as any other practical art, fort example. riding, or fencing;—that we are not secured from committing fallacies by such a classification of fallacies as logic supplies, as a rider would not be secured from falls by a classification of them;—and that the habit of attending to our mental processes while we are reasoning, rather interferes with than assists our reasoning well, as the horseman would ride worse rather than better, if he were to fix his attention upon his muscles when he is using them.

4. To this it may be added, that the peculiar habits which enable anyone to follow a *chain* of reasoning are excellently taught by mathematical study, and are hardly at all taught by logic. These habits consist in not only apprehending distinctly the demonstration of a single proposition when it is proved, but in retaining all the propositions thus proved, and using them in the ulterior steps of the argument with the same clear conviction, readiness, and familiarity, as if they were self-evident principles. Writers on Logic seldom give examples of reasoning in which several syllogisms follow each other; and they never give examples in which this progressive reasoning is so exemplified as to make the process familiar. Their chains generally consist only of two or three links. In Mathematics, on the contrary, every theorem is an example of such a chain; every proof consists of a series of assertions, of which each depends on the preceding, but of which the last inferences are no less evident or less easily applied than the simplest first principles. The language contains a constant succession of short and rapid references to what has been proved already; and it is justly assumed that each of these brief movements helps the reasoner forwards in a course of infallible certainty and security. Each of these hasty glances must possess the clearness of intuitive evidence, and the certainty of mature reflection; and yet must leave the reasoner's mind entirely free to turn instantly to the next point of his progress. The faculty of performing such mental processes well and readily is of great value, and is in no way fostered by the study of logic.

5. It is sometimes objected to the study of Mathematics as a discipline of reasoning, that it tends to render men insensible to all reasoning which is not

mathematical, and leads them to demand, in other subjects, proofs such as the subject does not admit of, or such as are not appropriate to the matter.

To this it may be replied, that these evil results, so far as they occur, arise either from the student pursuing too exclusively one particular line of mathematical study, or from erroneous notions of the nature of demonstration.

The present volume is intended to assist, in some measure, in remedying the too exclusive pursuit of one particular line of Mathematics, by shewing that the same simplicity and evidence which are seen in the Elements of Geometry may be introduced into the treatment of another subject of a kind very different; and it is hoped that we may thus bring the subject within the reach of those who cultivate the study of Mathematics as a discipline only. The remarks now offered to the reader are intended to aid him in forming a just judgment of the analogy between mathematical and other proof; which is to be done by pointing out the true grounds of the evidence of Geometry, and by exhibiting the views which are suggested by the extension of mathematical reasoning to sciences concerned about physical facts.

6. We shall therefore now proceed to make some remarks on the nature and principles of reasoning, especially as far as they are illustrated by the mathematical sciences.

Some of the leading principles which bear upon this subject are brought into view by the consideration of the question, “What is the foundation of the certainty arising from mathematical demonstration?” and in this question it is implied that mathematical demonstration is recognized as a kind of reasoning possessing a peculiar character and evidence, which make it a definite and instructive subject of consideration.

7. Perhaps the most obvious answer to the question respecting the conclusiveness of mathematical demonstration is this; —that the certainty of such demonstration arises from its being founded upon *Axioms*; and, conducted by steps, of which each might, if required, be stated as a rigorous *Syllogism.*

This answer might give rise to the further questions, What is the foundation of the conclusiveness of a Syllogism? and, What is the foundation of the certainty of an Axiom? And if we suppose the former enquiry to be left to Logic, as being the subject of that science, the latter question still remains to be considered. We may also remark upon this answer, that mathematical demonstration appears to depend upon Definitions, at least as much as upon Axioms. And thus, we are led to these questions: — Whether mathematical demonstration is founded upon Definitions, or upon Axioms, or upon both? and, What is the real nature of Definitions and of Axioms?

8. The question, What is the foundation of mathematical demonstration? was discussed at considerable length by Dugald Stewart*; and the opinion at which he arrived was, that the certainty of mathematical reasoning arises from its depending upon definitions. He expresses this further, by declaring that mathematical truth

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* *Elements of the Philosophy of the Human Mind*, Vol. 11.

is hypothetical, and must be understood as asserting only, that *if* the definitions are assumed, the conclusion follows. The same opinion has, I think, prevailed widely among other modern speculators on the same subject, especially among mathematicians themselves.

9. In opposition to this opinion, I urge, in the first place, that no one has yet been able to construct a system of mathematical truth by means of definitions alone, to the exclusion of axioms; although attempts having this tendency have been made constantly and earnestly. It is, for instance, well known to most readers, that many mathematicians have endeavored to get rid of Euclid's "Axioms" respecting straight lines and parallel lines; but that none of these essays has been generally considered satisfactory. If these axioms could be superseded, by definition or otherwise, it was conceived that the whole structure of Elementary Geometry would rest merely upon definitions; and it. was held by those who made such essays, that this would render the science more pure, simple, and homogeneous. If these attempts had succeeded, Stewart's doctrine might have required a further consideration; but it appears strange to assert that Geometry is supported by definitions, and not by axioms, when she cannot stir four steps without resting her foot upon an axiom.

10. But let us consider further the nature of these attempts to supersede the axioms above mentioned. They have usually consisted in endeavors so to frame the definitions, that these might hold the place which the axioms hold in Euclid's reasoning. Thus, the axiom, that "two straight lines cannot enclose a space," would be superfluous, if we were to take the following definition:—"A line is said to be *straight*, when *two* such lines cannot coincide in two points without coinciding *altogether*"

But when such a method of treating the subject is proposed, we are unavoidably led to ask,—whether it is allowable to lay down such a definition? It cannot be maintained that we may propound any form of words whatever as a definition, without any consideration whether or not it suggests to the mind any intelligible or possible conception. What would be said, for instance, if we were to state the following as a definition, "A line is said to be straight (or any other term) when two such lines cannot coincide in one point without coinciding altogether?" It would inevitably be remarked, that no such lines exist; or that such a property of lines cannot hold good without other conditions than those which this definition expresses; or, more generally, that the definition does not correspond to any conception which we can call up in our minds, and therefore can be of no use in our reasonings. And thus, it would appear, that a definition, to be admissible, must necessarily refer to and agree with some conception which we can distinctly frame in our thoughts.

11. This is obvious, also, by considering that the definition of a straight line could not be of any use, except we were entitled to apply it in the cases to which our geometrical propositions refer. No definition of straight lines could be employed in Geometry, unless it were in some way certain that the lines so defined are those by which angles are contained, those by which triangles are bounded, those of which parallelism may be predicated, and the like.

12. The same necessity for some general conception of such lines accompanying the definition, is implied in the terms of the definition above suggested. For what is there meant by "*such* lines?" Apparently, lines having some general character in which the property is necessarily involved. But how does it appear that lines may have such a character? And if it be self-evident that there may be such lines, this evidence is a necessary condition of this (or any equivalent) definition. And since this self-evident truth is the ground on which the course of reasoning must proceed, the simple and obvious method is, to state the property as a self-evident truth; that is, as an axiom. Similar remarks would apply to the other axiom above mentioned; and to any others which could be proposed on any subject of rigorous demonstration.

13. If it be conceded that such a conception accompanying the definition is necessary to justify it, we shall have made a step in our investigation of the grounds of mathematical evidence. But such an admission does not appear to be commonly contemplated by those who maintain that the conclusiveness of mathematical proof results from it depending on definitions. They generally appear to understand their tenet as if it implied arbitrary definitions. And something like this seems to be held by Stewart, when he says that mathematical truths are true *hypothetically*. Fur we understand by an hypothesis a supposition, not only which we may make, but may abstain from making, or may replace by a different supposition.

14. That the fundamental conceptions of Geometry are not arbitrary definitions, or selected hypotheses, will, I think, be clear to anyone who reasons geometrically at all. It is impossible to follow the steps of any single proposition of Geometry without conceiving a straight line and its properties, whether or not such a line be defined, and whether or not its properties be stated. That a straight line should be distinguished from all other lines, and that the axiom respecting it should be seen to be true, are circumstances indispensable to any clear thought on the subject of lines. Nor would it be possible to frame any coherent scheme of Geometry in which straight lines should be excluded, or their properties changed. Anyone who should make the attempt, would betray, in his first propositions, to all men who can reason geometrically, a reference to straight lines.

15. If, therefore, we say that Geometry depends on definitions, we must add, that they are *necessary*, not arbitrary definitions,—such definitions as we must have in our minds, so far as we have elements of reasoning at all. And the elementary hypotheses of Geometry, if they are to be so termed, are not hypotheses which are requisite to enable it to reach this or that conclusion; but hypotheses which are requisite for *any* exercise of our thoughts on such subjects.

16. Before I notice the bearing of this remark on the question of the necessity of axioms, I may observe that Stewart's disposition to consider definitions, and not axioms, as the true foundation of Geometry, appears to have resulted, in part, from an arbitrary selection of certain axioms, as specimens of all. He takes, as his examples, the axioms, "that if equals be added to equals the wholes are equal," that 'the whole is greater than its part;" and the like. If he had, instead of these, considered the more properly geometrical axioms,—such as those which I have mentioned; "that two straight lines cannot enclose a space;" or any of the axioms

which have been made the basis of the doctrine of parallels; for instance, Playfair's axiom, "that two straight lines which intersect each other cannot both of them be parallel to a third straight line;" it would have been impossible for him to have considered axioms as holding a different place from definitions in geometrical reasoning. For the properties of triangles are proved from the axiom respecting straight lines, as distinctly and directly, as the properties of angles are proved from the definition of a right angle. Of the many attempts made to prove the doctrine of parallels, almost all professedly, all really, assume some axiom or axioms which are the basis of the reasoning.

17. It is therefore very surprising that Stewart should so exclusively have fixed his attention upon the more general axioms, as to assert, following Locke, "that from [mathematical] Axioms it is not possible for human ingenuity to draw a single inference;"* and even to make this the ground of a contrast between geometrical Axioms and Definitions. The slightest examination of any treatise of Geometry might have shown him that there is no sense in which this can be asserted of Axioms, in which it is not equally true of Definitions; or rather, that while Euclid's Definition of a straight line leads to no truth whatever, his Axiom respecting straight lines is the foundation of the whole of Geometry; and that, though we can draw some inferences from the Definition of parallel straight lines, we strive in vain to complete the geometrical doctrine of such lines, without assuming some Axiom which enables us to prove the converse of our first propositions. Thus, that, which Stewart proposes as the distinctive character of Axioms, fails altogether; and with it, as I conceive, the whole of his doctrine respecting mathematical evidence.

18. That Geometry (and other Sciences when treated in a method equally rigorous) depends upon axioms as well as definitions, is supposed by the form in which it is commonly presented. And after what we have said, we shall assume this form to be a just representation of the real foundations of such sciences, till we can find a tenable distinction between axioms and definitions, in their nature, and in their use; and till we have before us a satisfactory system of Geometry without Axioms. And this system, we may remark, ought to include the Higher as well as the Elementary Geometry, before it can be held to prove that axioms are needless; for it will hardly be maintained, that the properties of circles depend upon definitions and hypotheses only, while those of ellipses require some additional foundation: that the comparison of curve lines requires axioms, while the relations of straight lines are independent of such principles.

19. Having then, I trust, cleared away the assertion, that mathematical reasoning rests ultimately upon definitions only, and that this is the ground of its peculiar cogency, I have to examine the real evidence of the truth of such axioms as are employed in the exact Mathematical Sciences. And we are, I think, already brought within view of the answer to this question. For if the definitions of Mathematics are not arbitrary, but necessary, and must, in order to be applicable in reasoning, be accompanied by a conception of the mind through which this necessity is seen; it is clear that this apprehension of the necessity of the properties

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* Elements of the Philosophy of the Human Mind Vol. 11. p. 38.

which we contemplate, is really the ground of our reasonings and the source of their irresistible evidence. And where clearly apprehend such necessary relations, it can make no difference whatever in the nature of our reasoning, whether we express them by means of definitions or of axioms. We define a straight line vaguely;—that it is that line which lies *evenly* between two, points: but we forthwith remedy this vagueness, by the axiom respecting straight lines: and thus we express our conception of a straight line, So far as is necessary for reasoning upon it. We might, in like manner, begin by defining a right-angle to be the angle made by a line which stands evenly between the two portions of another line; and we might add an axiom, that all right angles are equal. Instead of this, we define a right angle to be that which a line makes with another when the two angles on the two sides of it are equal. But in all these cases, we express our conception of a necessary relation of lines; and whether this be done in the form of definitions or axioms, is a matter of no importance.

20. But it may be asked, if it be thus unimportant whether we state our fundamental principles as axioms or definitions, why not reduce them all to definitions, and thus give to our system that aspect of independence which many would admire, and with which none need be displeased? And to this we answer, that if such a mode of treating the subject were attempted, our definitions would be so complex, and so obviously dependent on something not expressed, that they would be admired by none. We should have to put into each definition, as conditions, all the axioms which refer to the things defined. For instance, who would think it a gain to escape the difficulties of the doctrine of parallels by such a definition as this: "Parallel straight lines are those which being produced indefinitely both Ways do not meet; and which are such that if a straight line intersects one of them it must somewhere meet the other?" And in other cases, the accumulation of necessary properties would be still more cumbersome and more manifestly heterogeneous.

21. The reason of this difficulty is, that our fundamental conceptions of lines and other relations of space, are capable of being contemplated under several various aspects, and more than one of these aspects are needed in our reasonings.. We may take one such aspect of the conception for a definition; and then we must introduce the others by means of axioms. We may define parallels by their not meeting; but we must have some positive property, besides this negative one, in order to complete our reasonings respecting such lines. We have, in fact, our choice of several such self-evident properties, any of which we may employ for our purpose, as geometers well know; but with our naked definition, as they also know; we cannot proceed to the end. And in other cases, in like manner, our fundamental conception gives rise to various elementary truths, the connection of which is the basis of our reasonings: but this connection resides in our thoughts, and cannot be made to follow, as a logical result, from any assumed form of words, presented as a definition.

22. If it be further demanded, What is the nature of this bond in our thoughts by which various properties of lines are connected? perhaps the simplest answer is to say, that it resides in t*he idea of space*. We cannot conceive things in space

without being led to consider them as determined and related in some way or other to straight lines, right angles, and the like; and we cannot contemplate these determinations and relations distinctly, without assuming those properties of straight lines, of right angles, and of the rest, which are the basis of our Geometry. We cannot conceive or perceive objects at all, except as existing in space; we cannot contemplate them geometrically, without conceiving them in space which is subjected to geometrical conditions; and this mode of contemplation is, by language, analyzed into definitions, axioms, or both.

23. The truths thus seen and known, may be said to be known by *intuition*. In English writers this term has, of late, been vaguely used, to express all convictions which are arrived at without conscious reasoning, whether referring to relations among our primary perceptions, or to conceptions of the most derivative and complex nature. But if we were allowed to restrict the use of this term, we might conveniently confine it to those cases in which we necessarily apprehend relations of things truly, as soon as we conceive the objects distinctly. In this sense *axioms* may be said to be *known by intuition*; but this phraseology is not essential to our purpose.

24. It appears, then, that the evidence of the axioms of Geometry depends upon a distinct possession of the idea of space. These axioms are stated in the beginning of our Treatises, not as something which the reader is to learn, but as something which he already knows. No proof is offered of them; for they are the beginning not the end of demonstrations. The student's clear apprehension of the truth of these is a condition of the possibility of his pursuing the reasonings on which he is invited to enter.[*] Without this mental capacity, and the power of referring to it, in the reader, the writer's assertions and arguments are empty and unmeaning words; but then, this capacity and power are what all rational creatures alike possess, though habit may have developed it in very various degrees in different persons.

25. It has been common in the school of metaphysicians of which I have spoken, to describe some of the elementary convictions of our minds as

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[*] In this statement respecting the nature of Axioms, I find myself agreeing with the acute author of "Sematology." See the "Sequel to Sematology,"* p. 103. "An Axiom does not account for an intellection; it does but describe the requisite competency for it." It appears to me that this view is not familiar among English metaphysicians. I may here quote what I said at a former period, "However we may *define* force, it is necessary, in order to understand the elementary reasonings of this portion of science, that we should *conceive* it distinctly. Do we wish for a test of the distinctness of our conceptions? The test is, our being able to see the necessary truth of the Axioms on which our reasonings rest… These principles (the Axioms of Statics) are all perfectly evident as soon as we have formed the general conception of pressure but without that act of thought, they can have no evidence whatever given them by any form of words, or reference to other truths;—by definitions, or by illustrations from other kinds of quantity,—" *Thoughts on the Study of Mathematics*, p. 25.

fundamental laws of belief; and it appears to have been considered that this might be taken as a final and sufficient account of such convictions. I do not know whether any persons would be tempted to apply this formula, as a solution of our question rejecting the nature of axioms. If this were proposed, I should observe, that this form of expression seems to me, in such a case, highly unsatisfactory. For *laws* require and enjoin a conjunction of things which can be contemplated separately, and which would be disjoined if the law did not exist. It is a law of nature that terrestrial bodies, when free-fall downwards; for we can easily conceive such bodies divested of such a property. But we cannot say, in the same sense, that the impossibility of two straight lines inclosing a space arises from a law; for if they are straight lines, they need no law to compel this result. We cannot conceive straight lines exempt from such a law. To speak of this property as imposed by a law, is to convey an inadequate and erroneous notion of the close necessity, inviolable even in thought, by which the truth clings to the conception of the lines.

26. This expression of "laws of belief," appears to have found favor, on this account among others, that it recognizes a kind of analogy between the grounds of our reasoning on very abstract subjects, and the principles to which we have recourse in other cases when we manifestly derive our fundamental truths from facts, and when it is supposed to be the ultimate and satisfactory account of them to say, that they are laws of nature learnt by observation. But such an analogy can hardly be considered as a real recommendation by the metaphysician; since it consists in taking a case in which our knowledge is obviously imperfect and its grounds obscure, and in ending this case into an authority which shall direct the process and control the enquiry of a much more profound and penetrating kind of speculation. It cannot be doubted that we are likely to see the true grounds and evidence of our doctrines much more clearly in the case of Geometry and other rigorous systems of reasoning, than in collections of mere empirical knowledge, or of what is supposed to be such. It is both an unphilosophical and an indolent proceeding, to take the latter cases as a standard for the former.

27. I shall therefore consider it as Established, that in Geometry our reasoning depends upon axioms as well as definitions,—that the evidence of the truth of the axioms and of the propriety of the definitions resides in the idea of space,—and that the distinct possession of this idea, and the consequent apprehension of the Truth of the axioms which are its various aspects, is supposed in the student who is to pursue the path of geometrical reasoning. This being understood, I have little further to observe on the subject of Geometry. I will only remark—that all the conclusions which occur in the science follow purely from those first principles of which we have spoken;—that each proposition is rigorously proved from those which have been proved previously from such principles;—that this process of successive proof is termed *Deduction*;—and that the rules which secure the rigorous conclusiveness of each step are the rules of *Logic*, which I need not here dwell upon.

28. But I now proceed to consider some other questions to which our examination of the evidence of Geometry was intended to be preparatory;—How far do the statements hitherto made apply to other sciences? for instance, to such

sciences as are treated of in the present volume, Mechanics and Hydrostatic. To this I reply, that some such sciences at least, as for example the science of Statics, appear to me to rest on foundations exactly similar to Geometry:—that is to say, that they depend upon axioms,—self-evident principles, not derived in any immediate manner from experiment, but involved in the very nature of the conceptions which we must possess, in order to reason upon such subjects at all. The proof of this doctrine must consist of several steps, which I shall take in order.

29. In the first place, I say that the axioms of Statics are *self-evidently true*. In the beginning of the preceding Treatise, I have stated these barely as axioms, without addition or explanation, as the axioms of Geometry are stated in treatises on that subject. And such is the proper and orderly mode of exhibiting axioms; for, as has been said, they are to be understood as an expression of the condition of conception of the student. They are not to be learnt from without, but from within. They necessarily and immediately flow from the distinct possession of that idea, which if the student does not possess distinctly, all conclusive reasoning on the subject under notice is impossible. It is not the business of the deductive reasoner to communicate the apprehension of these truths, but to deduce others from them.

30. But though it may not be the author's business to elucidate the truth of the axioms as a deductive reasoner, it may still be desirable that he should do so as a philosophical teacher; and though it may not be possible to add anything to their evidence in the mind of him who possesses distinctly the idea from which they flow, it may be in our power to the beginner in obtaining distinct possession of this idea and unfolding it into its consequences. Accordingly, I have made some Remarks of this kind, tending to illustrate the self-evident nature of the "Axioms " of Statics and of Hydrostatics; and have inserted them in Book 1. and Book 2. respectively.

31. Some of the Axioms which are stated in Book 3, on the Laws of Motion, give occasion to remarks similar to those already made. "Thus Axiom 4, which asserts that if particles move in such a manner as always to preserve the same relative distances and positions, their motions will not be altered by supposing them rigidly connected, is evident by the same considerations as the Axioms concerning flexible and fluid bodies, already noticed in Book II. For the forces of rigidity are forces which would prevent a change of the distances and relative positions of the particles if there were a tendency to any such change; and if there be no such tendency, it makes no difference whether the potential resistance to be present or absent.

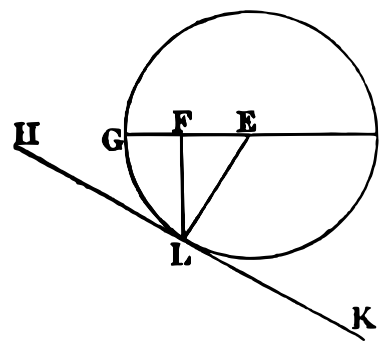
32. The 5th Axiom of Book in. which asserts that forces producing parallel and equal velocities at the same time, may be conceived to be, added; and the 6th Axiom, which asserts that in systems in motion the action and re-action are equal and opposite, are applications of what is stated in the second sentence of this third Book;—that the Definitions and Axioms of Statics are adopted and assumed in the case of bodies in motion. In the third Book, as in the first, forces are conceived as capable of addition, and matter is conceived as that which can resist force, and transmit it unaltered;

The 3rd, 8th, and 9th Axioms of Book III, like the 7th of Book II, are introduced to avoid the reasoning which depends on Limits.

33. In the case of Mechanics, as in the case of Geometry, the distinctness of the idea is necessary to a full apprehension of the truth of the axioms; and in the case of mechanical notions, it is far more common than in Geometry, that the axioms are imperfectly comprehended, in consequence of the want of distinctness and exactness in men's ideas. Indeed, this indistinctness of mechanical notions has not only prevailed in many individuals at all periods, but we can point out whole centuries, in which it has been, so far as we can trace, universal. And the consequence of this was, that the science of Statics, after being once established upon clear and sound principles, again fell into confusion, and was not understood as an exact science for two thousand years, from the time of Archimedes to that of Galileo and Stevinus.

34. In order 'to illustrate this indistinctness of mechanical ideas, I shall take from an ancient Greek writer an attempt to solve a mechanical problem; namely, the Problem of the Inclined Plane. The following is the mode in which Pappus professes* to answer this question:—"To find the force which will support a given weight *A* upon an inclined plane."

Let *HK* be the plane; let the weight *A* be formed into a sphere: let this sphere be placed in contact with the plane HK touching it in the point *L*, and let *E* be its center. Let *EG* be a horizontal radius, and *LF* a vertical line which meets it. Take a weight *B* which is to *A* as *EF* to *FG*. Then if *A* and *B* be suspended at *E* and *G* to the lever *EFG* of which the center of motion is *F*, they will balance; being supported, as it were, by the fulcrum *LF*. And the sphere, which is equal to the weight *A*, may be supposed to be collected at its center. If therefore B act at G, the weight A will be supported.



It may be observed that in this, attempt, the confusion of ideas is such, that the author assumes a weight which acts at *G*, perpendicularly on the lever *EFG*, and which is therefore a vertical force, as identical with a force which acts at G, to support the body in the inclined plane, and which is parallel to the plane.

35. When this kind of confusion was remedied, and when men again acquired distinct notions of pressure, and of the transmission of pressure from one point to another, the science of Statics was formed by Stevinus, Galileo, and their successors.*

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* Pappus, B. viii. Prop. ix. I purposely omit the confusion produced by this author's mode of treating the question, which he inquires the force which will draw a body up the inclined plane.

* See History of, the Inductive Sciences, B, vi. chap. I. sect. 2, On, the Revival of the Scientific Idea of Pressure.

The fundamental idea of Mechanics being thus acquired, and the requisite consequences of them stated in axioms, our reasonings proceed by the same 1 rigorous line of demonstration, and under the same logical rules as the reasonings of Geometry; and we have a science of Statics which is, like Geometry, an exact deductive science.

Section 2. On the Logic of Induction.

36. There are other portions of Mechanics which require to be considered in another manner; for in these there occur principles which are derived directly and professedly from experiment and observation. The derivation of principles by reasoning from facts is performed by a process which is termed *Induction*, which is very different from the process of *Deduction* already noticed, and of which we shall attempt to point out the character and method.

It has been usual to say of any general truths, established by the consideration and comparison of several facts, that they are obtained by *Induction*; but the distinctive character of this process has not been well pointed out, nor have any rules been laid down which may prescribe the form and ensure the validity of the process, as has been done for Deductive reasoning by common Logic. The *Logic of Induction* has not yet been constructed; a few remarks on this subject are all that can be offered here.

37. The Inductive Propositions, to which we shall here principally refer as examples of their class, are those elementary principles which occur in considering the motion of bodies, and of which some are called the Laws of Motion.[*] They are such as these;—a body not acted on by any force will move on forever uniformly in a straight line;—gravity is a uniform force;— if a body in motion be acted upon by any force, the effect of the force will be compounded with the previous motion;—when a body communicates motion to another directly, the momentum lost by the first body is equal to the momentum gained by the second. And I remark, in the first place, that in collecting such propositions from facts, there occurs a step corresponding to the term "Induction" (ἐπαγωγὴ inductio}. Some notion is *superinduced* upon the observed facts. In each inductive process, there is some general idea introduced, which is given, not by the phenomena, but by the mind. The conclusion is not contained in the premises, but includes them by the introduction of a new generality. In order to obtain our inference, we travel beyond the cases we have before us; we consider them as exemplifications of, or deviations from, some ideal case in which the relations are complete and intelligible. We take a standard, and measure the facts by it; and this standard is created by us, not offered by Nature. Thus, we assert, that a body left to itself will move on with unaltered velocity, not because our senses ever disclosed to us a body doing this, but because (taking this as our ideal case) we find that all actual cases are intelligible and explicable by mean of the notion of forces which cause change of

[*] Inductive Propositions in this work are, Book II. Propositions 25,26 32, 36, 37; Book III. Prop. 2, 3, 8,13.

motion, and which are exerted by surrounding bodies. In like manner, we see bodies striking each other, and thus moving, accelerating, retarding, and stopping each other; but in all this, we do not, by our senses, perceive that abstract quantity, *momentum*, which is always lost by one as it is gained by another. This momentum is a creation of the mind, brought in among the facts, in order to convert their apparent confusion into order, their seeming chance into certainty, their perplexing variety into simplicity. This the idea of *momentum gained and lost* does; and, in like manner in any other case in which inductive truths are established, some idea is introduced, as the means of passing from the facts to the truth.

38. The process of mind of which we here speak can only be described by suggestion and comparison. One of the most common of such comparisons, especially since the time of Bacon, is that which speaks of induction as the *interpretation* of facts. Such an expression is appropriate; and it may easily be seen that it includes the circumstance which we are now noticing;—the superinduction of an idea upon the facts by the interpreting mind. For when we read a page, we have before our eyes only black and white, form and color; but by an act of the mind, we transform these perceptions into thought and emotion. The letters are nothing of themselves; they contain no truth, if the mind does not contribute its share: for instance, if we do not know the language in which the words are written. And if we are imperfectly acquainted with the language, we become very clearly aware how much a certain activity of the mind is requisite in order to convert the words into propositions, by the extreme effort which the business of interpretation requires. Induction, then, may be conveniently described as the *interpretation of phenomena*.

39. But I observe further, that in thus inferring truths from facts, it is not only necessary that the mind should contribute to the task its own idea, but, in order that the propositions thus obtained may have any exact import and scientific value, it is requisite that the idea be perfectly *distinct* and precise. If it be possible to obtain some vague apprehension of truths, while the ideas in which they are expressed remain indistinct and ill-defined, such knowledge cannot he available for the purposes we here contemplate. In order to construct a science, all our fundamental ideas must be distinct; and among them, those which Induction introduces.

40. This necessity for distinctness in the ideas which we employ in Induction, makes it proper to *define*, in a precise and exact manner, each idea when it is thus brought forwards. Thus, in establishing the propositions which we have stated as our examples in these cases, we have to define *force* in general; *uniform force*; *compounding* of motions; *momentum*. The construction of these definitions is an essential part of the process of Induction, no less than the assertion of the inductive truth itself.

41. But in order to justify and establish the inference which we make, the ideas which we introduce must not only be distinct, but also *appropriate*. They, must be exactly and closely applicable to the facts; so that when the idea is in our possession, and the facts under our notice, we perceive that the former includes and takes up the latter. The idea is only a more precise mode of apprehending the

facts, and it is empty and unmeaning if it be anything else; but if it be thus applicable, the proposition which is asserted by means of it is true, precisely because the facts *are* facts. When we have defined *force* to be *the cause of change of motion*, we see that, as we remove *external* forces, we do, in actual experiments, remove all the change of motion; and therefore the, proposition that there is in bodies no *internal* cause of change of motion, is true. When we have defined *momentum* to be the *product of the velocity and quantity of matter*, we see that in the actions of bodies, the *effect* increases as the *momentum* increases; and by measurement, we find that the effect may consistently be *measured* by the momentum. The ideas here employed are not only distinct in the mind, but applicable in the world; they are the elements, not only of relations of thought, but of laws of nature.

42. Thus an inductive inference requires an idea from within, facts from without, and a coincidence of the two. The idea must be distinct, otherwise we obtain no scientific truth; it must be appropriate, otherwise the facts cannot be steadily contemplated by means of it; and when they are so contemplated, the Inductive Proposition must be seen to be verified by the evidence of sense.

It appears from what has been said, that in establishing a proposition by Induction, the *definition* of the *idea* and the *assertion* of the *truth*, are not only both requisite, but they are correlative. Each of the two steps contains the verification and justification of the other. The proposition derives its meaning from the definition; the definition derives its reality from the proposition. If they are separated, the definition is arbitrary or empty, the proposition is vague or verbal.

43, Hence we gather, that in the Inductive Sciences, our Definitions and our Elementary Inductive Truths ought to be introduced together. There is no value or meaning in definitions, except with reference to the truths which they are to express. Discussions about the definitions of any science, taken separately, cannot therefore be profitable, if the discussion do not refer, tacitly or expressly, to the fundamental truths of the science; and in all such discussions, it should be stated what are taken as the fundamental truths. With such a reference to Elementary Inductive Truths clearly understood, the discussion of Definitions may be the best method of arriving at that clearness of thought, and that arrangement of facts, which Induction requires.

I will now note some of the differences which exist between Inductive and Deductive Reasoning, in the modes in which they are presented,

44. One leading difference in these two kinds of reasoning is, that in Deduction we infer particular from general truths; in Induction, on the contrary, we infer general from particular. Deductive proof consists of many steps, in each of which we apply known general propositions in particular cases;—“all triangles have their angles equal to two right angles, therefore this triangle has; therefore, etc.” In Induction, on the other hand, we have a single step in which we pass from many particular Propositions to-one general proposition; “This stone falls downwards; so, do those others;—all stones fall downwards.” And the former inference flows necessarily from the relation of general and particular; but the latter, as we have

seen, derives its power of convincing from the introduction of a new idea, which is distinct and appropriate, and which supplies that generality which the particulars cannot themselves offer.

45. I observe also that this difference of process in inductive and deductive proofs, may be most properly marked by a difference in the form in which they are stated. In Deduction, the *Definition* stands at the beginning of the proposition; in Induction, it may most suitably stand at or near the end. Thus, the definition of a uniform force is introduced in the course of the proposition that gravity is a uniform force. And this arrangement represents truly the real order of proof; for, historically speaking, it was taken for granted that gravity was a uniform force; but the question remained, what was the right definition of a uniform force. And in the establishment of other inductive principles, in like manner, definitions cannot be laid down for any useful purpose, till we know the propositions in which they are to be used. They may therefore properly come each at the conclusion of its corresponding proposition.

46. The ideas and definitions to which we are thus led by our inductive process, may bring with them Axioms. Such Axioms may be self-evident as soon as the inductive idea has been distinctly apprehended, in the same manner as was explained respecting the fundamental ideas of Geometry and Statics. And thus *Axioms*, as well as Definitions, may come at the end of our Inductive, Propositions; and they thus assume their proper place at the beginning of the deductive propositions which follow them, and are proved from them. Thus, in Book 3, Axioms 8 and 9, come after the definition of Accelerating Force, and stand between Propositions 13 and 14.

47. Another peculiarity in inductive reasoning may be noticed. In a deductive demonstration, the reference is always to what has been already proved; in establishing an Inductive Principle, it is most convenient that the reference should be to subsequent propositions. For the proof of the Inductive Principle consists in this;—that the principle being adopted, consequences follow which agree with fact; but the demonstration of these consequences may require many steps, and several special propositions. Thus, the Inductive Principle, that gravity is a uniform force, is established by shewing that the law of descent, which falling bodies follow in fact, is explained by means of this principle; namely, the law that the distance is as the square of the time from the beginning of the motion. But the proof of such a property, from the definition of a uniform force, requires many steps, as may be seen in the preceding Treatise, Book 3. Proposition 5: and this proof must be referred to, along with several others, in order to establish the truth, that gravity is a uniform force.

48. It may be suggested, that, this being the case, the propositions might be transposed, so that the inductive proof might come after those propositions to which it refers. But if this were done, all the propositions which depend upon the laws of motion must be proved hypothetically only. For instance, we must say, "*If*, in the communication of motion, the momentum lost and gained be equal, the velocity acquired by a body falling down an inclined plane, will be equal to that acquired by falling down the height." This would be inconvenient, and even if it

were done, that completeness in the line of demonstration which is the object of the change, could not be obtained; for the transition from the particular cases to the general truth, which must occur in the Inductive Proposition, could not be in any way justified according to rules of Deductive Logic.

I have, therefore, in the preceding pages, placed the Inductive Principle first in each line of reasoning, and have ranged after it the Deductions from it, which justify and establish it, as their first office, but which are more important as its consequences and applications, after, it is supposed to be established.

49. I have used one common *formula* in presenting the proof of each of the Inductive Principles which I have introduced; namely, after stating or exemplifying the facts which the induction includes, I have added "These results can be clearly explained and rigorously deduced by introducing the *Idea* or the *Definition*" which belongs to each case, "and the *Principle*" which expresses the inductive truth. I do not mean to assert that this formula is the only right one, or even the best; but it appears to me to bring under notice the main circumstances which render an induction systematic and valid.

50. It may be observed, however, that this formula does not express the full cogency of the proof. It declares only that the results *can* be clearly explained and rigorously deduced by the employment of a certain definition and a certain proposition. But in order to make the conclusion demonstrative, we ought to be able to declare that the results can be clearly explained and rigorously deduced *only* by the definition and proposition which we adopt. And, *in* reality, the mathematician's conviction of the truth of the Laws of Motion does depend upon his seeing that they (or laws equivalent to them) afford the *only* means of clearly expressing and deducing the actual facts. But this conviction, that no other law than those proposed can account for the known facts, finds its place in the mind gradually, as the contemplation of the consequences of the law and the various relations of the facts becomes steady and familiar. I have therefore not thought it proper to require such a conviction along with the first assent to the inductive truths which I have here stated.

51. The propositions established by Induction are termed *Principles*, because they are the starting points of trains of deductive reasoning. In the system of deduction, they occupy the same place as axioms; and accordingly, they are termed so by Newton—"Axiomata sive leges motus." Stewart objects strongly to this expression:* and it would be difficult to justify it; although to draw the line between Axioms and inductive principles may be a harder task than at first appears.

52. But from the consideration that our Inductive Propositions are the principles or beginnings of our deductive reasoning, and so far at least stand in the place of axioms, we may gather *this* lesson,—that they are not to be multiplied without necessity. For instance, if in a treatise on Hydrostatics, we should state as separate propositions, that "air has weight;" and that "the mercury in the barometer by the weight of the air:" and should prove both the one and the other

* Elem. Phil. Human Mind. Vol. if. p. 44.

by reference to experiment; we should offend against the maxims of Logic. These propositions are connected; the latter may be demonstrated deductively from the former; the former may be inferred inductively from the facts which prove the latter. One of these two courses ought to be adopted; we ought not to have two ends of our reasoning up-wards, or two beginnings of our reasoning downwards.

53. I shall not now extend these Remarks further. They may appear to many barren and unprofitable speculations; but those who are familiar with such subjects, will perhaps find in them something which, if well founded, is not without some novelty for the English reader. Such will, I think, be the case, if I have satisfied him,—that mathematical truth depends on Axioms as well as Definitions,—that the evidence of geometrical Axions is to be found only in the distinct possession of the Idea of Space,—that other branches of mathematics also depend on Axioms,—and that the evidence of these Axioms is to be sought in some appropriate Idea;—that the evidence of the Axioms of Statics, for instance, resides in the Ideas of Force and Matter that in the process of Induction the mind must supply an Idea in addition to the Facts apprehended by the senses;—that in each such process we must introduce one or more Definitions, as well as a Proposition;—that the Definition and the Proposition are correlative, neither being useful or valid without the other;—and that the Formula of inductive reasoning must be in many respects the reverse of the common logical formulae of deduction.

THE END

THE

MECHANICAL EUCLID,

CONTAINING THE

LEMENTS OF MECHANICS AND HYDROSTATICS

DEMONSTRATED AFTER THE MANNER OF

THE ELEMENTS OF GEOMETRY;

AND INCLUDING THE

PROPOSITIONS FIXED UPON BY THE UNIVERSITY OF CAMBRIDGE AS REQUISITE FOR THE DEGREE OF B.A.

TO WHICH ARE ADDED

REMARKS ON MATHEMATICAL REASONING

AND ON

THE LOGIC OF INDUCTION.

BY THE REV. WILLIAM WHEWELL, M.A.

FELLOW AND TUTOR OF TRINITY COLLEGE.

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PREFACE.

By calling this little work The Mechanical Euclid, I mean to imply, that I have aimed at making it such a coherent system of exact reasoning, as that for which Euclid's name is become a synonym. Such a system of Mechanics, when once constructed, can hardly fail to be of use in that disciplinal employment of Mathematics in which Euclid's Elements of Geometry have hitherto most deservedly held their place without a rival. And such an application of the elementary portions of Mechanics and Hydrostatics having been resolved upon by the University of Cambridge, and having been appointed as an essential part of the examinations for the usual degrees, I have very gladly made my attempt to produce such a Manual as this occasion seems to require. It is proper to state, however, that though I had the honour to be one of the members of the Syndicate which drew up the Report recommending this change in the University examinations, (a recommendation adopted by the Senate), I am not in any degree authorized to put forth this book in any public capacity. The responsibility for everything which it contains, both as to plan and execution, rests upon me as an individual. The Treatise has no claim to adoption except what depends upon

its own merits, and its consistency with the decrees and usages of the University in respect to examinations.

Although the work is now published with immediate reference to the new scheme of examination, it will easily be seen that it does not refer solely to that object. I have introduced many propositions into my series, which do not occur in the list of Propositions published by the the University as requisite for a degree. This I have done, partly in order to make the proofs rigorous and the propositions of more convenient length and form; and partly in order to afford the means of extending this line of examination hereafter, if it should be thought desirable to do so. For the latter purpose I have also added to the two Books on Statics and on Hydrostatics, a third Book on the Laws of Motion.

The cases in Mechanics in which fundamental principles are proved by reference to facts, appeared to me to afford a favourable opportunity of giving, if possible, greater precision to the phrase, so commonly employed, of *Reasoning by Induction*. In order to mark as distinctly as I could the nature of this reasoning, I have reduced all the proofs of this kind which occur in the resent work, to one common shape or *formula*. And in the Remarks at the end of the work, I have endeavoured to point out the general applicability of this formula, the conditions on which its conclusiveness depends, and the lessons which it suggests. If this attempt to draw the outline of a

system of *Inductive Logic*, different from the common Syllogistic or Deductive Logic, be in any degree successful, it must, I think, be considered as an approximation to the solution of a very prominent and important problem.

The Remarks to which I have just referred contain moreover some reflections upon the use of Mathematics, especially *mixed* Mathematics, as an instrument of education; and also some observations upon the grounds of Mathematical Reasoning. These observations are closely connected with the views here presented respecting the peculiarities of Inductive Reasoning. I should wish the remarks now offered to be taken in connexion with those which I published a little while ago, under the title of "Thoughts on the Study of Mathematics, as a Part of a Liberal Education;" a pamphlet in which I recommended a change in the examinations of the University, such as has now been adopted. The whole subject of the grounds of the truth of our mechanical doctrines will, I hope, be found to derive much illustration from the History of Mechanics contained in my "History of the Inductive Sciences."

In my remarks on Mathematical Reasoning, I have not hesitated to dissent from the views propounded by the late Professor Dugald Stewart, and I have stated my dissent and the reasons for it without ceremony. I am persuaded that no one who is solicitous about truth in such matters, will see in this course any want of respect for that amiable and instructive writer.

The wish to put the elementary portions of Mechanics in a form in which they might be extensively studied in Universities, led me, a few years ago, to publish a little work under the title of "The First Principles of Mechanics, with Historical and Practical Illustrations." But the effect of mixed Mathematics as a discipline is, I conceive, likely to be far better answered by the more rigorous scheme of Mechanics which the University has sanctioned, than by such a treatment of the subject as was then presented; and the Historical Illustrations, which appeared to excite some interest in the public, are given much more completely in the History to which I have just referred. I shall therefore consider these "First Principles" as now superseded, and shall not republish the work. The Practical Illustrations may be perhaps incorporated in some future publication in an improved form.

I have prefixed to the Mechanical part of the present volume an Introduction, in which I have given those propositions of Pure Mathematics which are requisite in the course of my reasoning. The Algebraical part of this Introduction is taken, with little alteration, from the well-established Treatise of Dr Wood, by the kind permission of the Author.

The Propositions fixed upon by the University as requisite for the degree of Bachelor of Arts are marked with an asterisk, thus, * Prop. II.

Trinity College, *April* 11, 1837.

INTRODUCTION.

ELEMENTARY PURE MATHEMATICS.

ALGEBRA.

*(1.) *To define and explain Algebraical Signs.*

ART. 1. THE method of representing the relation of abstract quantities by letters and characters, which are made the signs of such quantities and their relations, is called ALGEBRA.

Known or determined quantities are usually represented by the first letters of the alphabet a, b, c, d, &c. and unknown or undetermined quantities by the last y, x, w, &c.

The following signs are made use of to express the relations which the quantities bear to each other.

2. $+$ *Plus*, signifies that the quantity to which it is prefixed must be added. Thus $a + b$ signifies that the quantity represented by b is to be added to the quantity represented by a; if a represent 5, and b, 7, then $a + b$ represents 12.

If no sign be placed before a quantity, the sign $+$ is understood. Thus a signifies $+ a$. Such quantities are called positive quantities.

3. $-$ *Minus*, signifies that the quantity to which it is prefixed must be subtracted. Thus, $a - b$ signifies that b must be taken from a; if a be 7, and b, 5, $a - b$ expresses 7 diminished by 5, or 2.

Quantities to which the sign $-$ is prefixed are called negative quantities.

4. $\times$ *Into*, signifies that the quantities between which it stands are to be multiplied together. Thus $a \times b$ signifies that the quantity represented by a is to be multiplied by the quantity represented by b*.

This sign is frequently omitted; thus $a\,b\,c$ signifies $a \times b \times c$, or a full point is used instead of it; thus $1 \times 2 \times 3$, and $1 \,.\, 2 \,.\, 3$ signify the same thing.

5. If in multiplication the same quantity be repeated any number of times, the product is usually expressed by placing above the quantity the number which represents how often it is repeated; thus a, $a \times a$, $a \times a \times a$, $a \times a \times a \times a$, and a^1, a^2, a^3, a^4, have respectively the same signification. These quantities are called *powers*; thus a^1, is called the *first power* of a; a^2, the *second power*, or *square* of a; a^3, the *third power*, or *cube* of a; a^4, the *fourth power*, or *biquadrate* of a. The succeeding powers have no names in common use except those which are expressed by means of number; thus a^7 is the *seventh power* of a, or *a to the seventh power*; and a^n is a to the n^{th} power.

The numbers 1, 2, 3, &c. are called the *indices* of a; or *exponents* of the powers of a.

6. $\div$ *Divided by*, signifies that the former of the quantities between which it is placed is to be divided by the latter. Thus, $a \div b$ signifies that the quantity a is to be divided by b.

The division of one quantity by another is frequently represented by placing the dividend over the divisor with a line between them, in which case the expression is called *a fraction*. Thus, $\frac{a}{b}$ signifies a

* By quantities, we understand such magnitudes as can be represented by numbers; we may therefore without impropriety speak of the multiplication, division, &c. of quantities by each other.

divided by b; and a is the numerator, and b the denominator of the fraction; also $\frac{a+b+c}{e+f+g}$ signifies that a, b, and c added together, are to be divided by e, f, and g added together.

7. A quantity in the denominator of a fraction is also expressed by placing it in the numerator, and prefixing the negative sign to its index; thus a^{-1}, a^{-2}, a^{-3}, a^{-n} signify $\frac{1}{a^1}$, $\frac{1}{a^2}$, $\frac{1}{a^3}$, $\frac{1}{a^n}$ respectively; these are called the *negative powers* of a.

8. The *reciprocal* of a fraction is the fraction inverted. Thus $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$; and $\frac{1}{a}$ is the reciprocal of a.

9. A line drawn over several quantities signifies that they are to be taken collectively, and it is called a *vinculum*. Thus $\overline{a-b+c} \times \overline{d-e}$ signifies that the quantity represented by $a-b+c$ is to be multiplied by the quantity represented by $d-e$. Let a stand for 6; b, 5; c, 4; d, 3; and e, 1; then $a-b+c$ is $6-5+4$, or 5; and $d-e$ is $3-1$, or 2; therefore $\overline{a-b+c} \times \overline{d-e}$ is 5×2, or 10. $\overline{ab-cd} \times \overline{ab-cd}$ or $\overline{ab-cd}|^2$ signifies that the quantity represented by $ab-cd$ is to be multiplied by itself.

Instead of a line, brackets are sometimes used, as $(ab-cd)^2$, $\{a-b+c\} . \{d-e\}$.

10. $=$ *Equal to*, signifies that the quantities between which it is placed are equal to each other, thus $ax-by=cd+ad$, signifies that the quantity $ax-by$ is equal to the quantity $cd+ad$.

11. The *square root* of any proposed quantity is that quantity whose square, or second power, gives the proposed quantity. The *cube root,* is that quantity whose cube gives the proposed quantity, &c.

The signs $\sqrt{}$, or $\sqrt[2]{}$, $\sqrt[3]{}$, $\sqrt[4]{}$, &c. are used to express the square, cube, biquadrate, &c. roots of the quantities before which they are placed.

$$\sqrt{a^2} = a, \quad \sqrt[3]{a^3} = a, \quad \sqrt{a^4} = a, \text{ \&c.}$$

These roots are all represented by the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. placed a little above the quantities, to the right. Thus $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, $a^{\frac{1}{n}}$, represent the square, cube, fourth and n^{th} root of a, respectively; $a^{\frac{5}{2}}$, $a^{\frac{7}{3}}$, $a^{\frac{3}{5}}$, represent the square root of the fifth power, the cube root of the seventh power, the fifth root of the cube of a.

12. If these roots cannot be exactly determined, the quantities are called *irrational* or *surds.*

13. Points are made use of to denote *proportion,* thus $a : b :: c : d$, signifies that a bears the same proportion to b that c bears to d.

14. The number prefixed to any quantity, and which shews how often it is to be taken, is called its *coefficient.* Thus, in the quantities $7ax$, $6by$, and $3dz$, 7, 6, and 3 are called the coefficients of ax, by, and dz respectively.

When no number is prefixed, the quantity is to be taken once, or the coefficient 1 is understood.

These numbers are sometimes represented by letters, which are called coefficients.

15. Similar, or *like* algebraical quantities are such as differ only in their coefficients; $4a$, $6ab$, $9a^2$,

$3a^2bc$, are respectively similar to $15a$, $3ab$, $12a^2$, $15a^2bc$, &c.

Unlike quantities are different combinations of letters; thus, ab, a^2b, ab^2, abc, &c. are unlike.

16. A quantity is said to be a *multiple* of another, when it contains it a certain number of times exactly: thus $16a$ is a multiple of $4a$, as it contains it exactly four times.

17. A quantity is called a *measure* of another, when the former is contained in the latter a certain number of times exactly; thus, $4a$ is a measure of $16a$.

18. When two numbers have no *common measure* but unity, they are said to be *prime* to each other.

19. A *simple* algebraical quantity is one which consists of a single term, as a^2bc.

20. A *binomial* is a quantity consisting of two terms, as $a + b$, or $2a - 3bx$. A *trinomial* is a quantity consisting of three terms, as $2a + bd + 3c$.

21. The following examples will serve to illustrate the method of representing quantities algebraically:—

Let $a = 8$, $b = 7$, $c = 6$, $d = 5$ and $e = 1$; then

$$3a - 2b + 4c - e = 24 - 14 + 24 - 1 = 33.$$

$$ab + ce - bd = 56 + 6 - 35 = 27$$

$$\frac{a+b}{c-e} + \frac{3b-2c}{a-d} = \frac{8+7}{6-1} + \frac{21-12}{8-5}$$

$$= \frac{15}{5} + \frac{9}{3} = 6.$$

$$d^2 \times \overline{a - c} - 3ce^2 + d^3 = 25 \times 2 - 18 + 125$$

$$= 50 - 18 + 125 = 157.$$

*(2.) *To add and subtract simple Algebraical Quantities.*

22. The addition of algebraical quantities is performed by connecting those that are *unlike* with their proper signs, and collecting those that are *similar* into one sum.

Examples:

$$\begin{array}{rr}
\text{Add} & \text{Add} \\
4x & 5ax \\
3x & -ax \\
7a & by \\
-2a & -cy \\
\hline
\text{Sum } 7x+5a & \text{Sum } 4ax+by-cy
\end{array}$$

$$\begin{array}{rr}
a+2bx-y^2 & a+3b \\
b-bx+3y^2 & a+\overline{n-4}\,b \\
\hline
\text{Sum } a+b+bx+2y^2 & \text{Sum } 2a+\overline{n-1}\,b
\end{array}$$

23. Subtraction, or the taking away of one quantity from another, is performed by changing the sign of the quantity to be subtracted, and then adding it to the other by the rules laid down in Art. 22.

$$\begin{array}{llll}
\text{From} & 7x & \text{From} & 7x+3a \\
\text{Subtract} & x & \text{Subtract} & 5a-x \\
\hline
\text{Diff.} & 7x-x \text{ or } 6x & \text{Diff.} & 7x+x+5a-3a \\
 & & & \text{or } 8x+2a
\end{array}$$

$$\begin{array}{ll}
\text{From} & 4x^2+5ax-y^2 \\
\text{Subtract} & 3x^2-3ax+y^2 \\
\hline
\text{Diff.} & x^2+13ax-2y^2
\end{array}$$

* (3.) *To multiply simple Algebraical Quantities.*

24. The multiplication of simple algebraical quantities must be represented according to the notation pointed out in Art. 4 and 5. Thus, $a \times b$, or $a\,b$, represents the product of a multiplied by b; $a\,b\,c$, the product of the three quantities a, b, and c.

It is also indifferent in what order they are placed, $a \times b$ and $b \times a$ being equal.

25. If the quantities to be multiplied have coefficients, these must be multiplied together as in common arithmetic; the literal product being determined by the preceding rules.

Thus, $3a \times 5b = 15ab$; because

$$3 \times a \times 5 \times b = 3 \times 5 \times a \times b = 15ab.$$

26. The powers of the same quantity are multiplied together by adding the indices: thus, $a^2 \times a^3 = a^5$; for $aa \times aaa = aaaaa$. In the same manner,

$$a^m \times a^n = a^{m+n}; \text{ and } 3a^2x^3 \times 5\,axy^2$$
$$= 15a^3x^4y^2.$$

27. If the multiplier or multiplicand consist of several terms, each term of the latter must be multiplied by every term of the former, and the sum of all the products taken, for the whole product of the two quantities.

* (4.) *To divide simple Algebraical Quantities.*

28. To divide one quantity by another, is to determine how often the latter is contained in the former, or what quantity multiplied by the latter will produce the former.

Thus, to divide $a\,b$ by a is to determine how often a must be taken to make up ab; that is, what quantity multiplied by a will give ab; which we know

is b. From this consideration are derived all the rules for the division of algebraical quantities.

If only a part of the product which forms the divisor be contained in the dividend, the division must be represented according to the direction in Art. 6, and the quantities contained both in the divisor and dividend expunged.

Thus $15a^2b^2c$ divided by $3a^2bx$ is $\frac{15a^2b^2c}{3a^2bx}$, which is equal to $\frac{5bc}{x}$; expunging from the dividend and from the divisor the quantities 3, a^2, and b.

* (5.) *To reduce Fractions to others of equal value which have a common denominator.*

29. Fractions are changed to others of equal value with a common denominator, by multiplying each numerator by every denominator except its own, for the new numerator; and all the denominators together for the common denominator.

Let $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, be the proposed fractions; then $\frac{adf}{bdf}$, $\frac{cbf}{bdf}$, $\frac{edb}{bdf}$, are fractions of the same value with the former, having the common denominator bdf.

For $\frac{adf}{bdf}=\frac{a}{b}$; $\frac{cbf}{bdf}=\frac{c}{d}$; and $\frac{edb}{bdf}=\frac{e}{f}$ (Art. 28); the numerator and denominator of each fraction having been multiplied by the same quantity viz.—the product of the denominators of all the other fractions.

30. When the denominators of the proposed fractions are not prime to each other, find their greatest common measure; multiply both the numerator and

denominator of each fraction by the denominators of all the rest, divided respectively by their greatest common measure; and the fractions will be reduced to a common denominator in lower terms* than they would have been by proceeding according to the former rule.

Thus $\frac{a}{mx}$, $\frac{b}{my}$, $\frac{c}{mz}$, reduced to a common denominator are $\frac{ayz}{mxyz}$; $\frac{bxz}{mxyz}$; $\frac{cxy}{mxyz}$.

* (6). *To add together simple Algebraical Fractions.*

31. If the fractions to be added have a common denominator their sum is found by adding the numerators together and retaining the common denominator. Thus,

$$\frac{2a}{5} + \frac{a}{5} = \frac{3a}{5}$$

$$\frac{a + 2x}{3} + \frac{a - x}{3} = \frac{2a}{3}$$

$$\frac{7x + y}{a} + \frac{2y - 5x}{a} = \frac{2x - 4y}{a}.$$

32. If the fractions have not a common denominator, they must be transformed to others of the same value which have a common denominator, (by Art. 29), and then the addition may take place as before. Thus,

$$\frac{a}{3} + \frac{a}{5} = \frac{5a}{15} + \frac{3a}{15} = \frac{8a}{15}$$

$$\frac{a}{b} + \frac{a}{x} = \frac{ax}{6x} + \frac{ab}{6x} = \frac{ax + ab}{6x}$$

* To obtain them in the *lowest* terms, each must be reduced to another of equal value, with the denominator which is the least common multiple of all the denominators.

$$\frac{a}{b} + 1 = \frac{a}{b} + \frac{b}{b} = \frac{a+b}{b};$$

$$2 - \frac{a}{3x} = \frac{6x - a}{3x}.$$

* (7.) *To multiply simple Algebraical Fractions.*

33. To multiply a fraction by any quantity, multiply the numerator by that quantity and retain the denominator.

Thus $\frac{a}{b} \times c = \frac{ac}{b}$. For if the quantity to be divided be c times as great as before, and the divisor the same, the quotient must be c times as great.

34. The product of two fractions is found by multiplying the numerators together for a new numerator, and the denominators for a new denominator.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be the two fractions: then $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

For if $\frac{a}{b} = x$, and $\frac{c}{d} = y$, by multiplying the equal quantities $\frac{a}{b}$ and x by b, $a = bx$ (Art. 28), in the same manner $c = dy$; therefore, by the same axiom, $ac = bdxy$; dividing these equal quantities, ac and $bdxy$ by bd, we have $\frac{ac}{bd} = xy = \frac{a}{b} \times \frac{c}{d}$.

* (8). *To divide simple Algebraical Fractions.*

35. To divide a fraction by any quantity, multiply the denominator by that quantity, and retain the numerator.

The fraction $\frac{a}{b}$ divided by c, is $\frac{a}{bc}$. Because $\frac{a}{b} = \frac{ac}{bc}$, and a c^{th} part of this is $\frac{a}{bc}$; the quantity to be divided being a c^{th} part of what it was before, and the divisor the same.

36. To divide a quantity by any fraction, multiply the quantity by the reciprocal of the fraction. (Art. 8.)

If we divide c by $\frac{a}{b}$ we obtain $\frac{bc}{a}$. For if

$$c \div \frac{a}{b} = x, \quad c = x \times \frac{a}{b}, \text{ or } c = \frac{ax}{b}, \text{ and } x = \frac{bc}{a}.$$

* (9). *Algebraical definition of Proportion.*

37. Four quantities are said to be proportionals, when the first is the same multiple, part, or parts of the second, that the third is of the fourth.

Thus the four quantities 8, 12, 6, 9, are proportionals; for 8 is $\frac{2}{3}$ of 12, and 6 is $\frac{2}{3}$ of 9.

In this case $\frac{8}{12} = \frac{6}{9}$; and generally a, b, c, d are proportionals if $\frac{a}{b} = \frac{c}{d}$. This is usually expressed by saying a is to b, as c to d; and thus represented, $a : b :: c : d$.

The terms a and d are called the *extremes*, and b and c the *means*.

The fraction $\frac{a}{b}$ is called the *ratio* of a to b.

* (10). *Algebraical consequences of Proportion.*

38. When $\frac{a}{b} = \frac{c}{d}$, if a be equal to b, c is equal to d, and if a be less than b, c is less than d, and if a be greater than b, c is greater than d.

39. When four quantities are proportionals, the product of the extremes is equal to the product of the means.

Let a, b, c, d be the four quantities; then, since they are proportionals, $\frac{a}{b} = \frac{c}{d}$; and by multiplying both sides by bd, $ad = bc$.

Any three terms in a proportion $a : b :: c : d$ being given, the fourth may be determined from the equation $ad = bc$.

40. If the first be to the second as the second to the third, the product of the extremes is equal to the square of the mean.

For (Art. 39) if $a : x :: x : b$, $ab = x^2$.

41. If the product of two quantities be equal to the product of two others, the four are proportionals, making the terms of one product the means, and the terms of the other the extremes.

Let $xy = ab$, then dividing by ay, $\frac{x}{a} = \frac{b}{y}$,

or, $x : a :: b : y$.

42. If $a : b :: c : d$, and $c : d :: e : f$, then will $a : b :: e : f$.

Because $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$, therefore $\frac{a}{b} = \frac{e}{f}$; or

$a : b :: e : f$.

43. If four quantities be proportionals, they are also proportionals when taken *inversely*.

If $a : b :: c : d$, then $b : a :: d : c$. For $\frac{a}{b} = \frac{c}{d}$, and dividing unity by each of these equal

quantities; or taking their reciprocals, $\frac{b}{a} = \frac{d}{c}$; (Art. 36)

that is, $b : a :: d : c$.

44. If four quantities be proportionals they are proportionals when taken *alternately*.

If $a : b :: c : d$, then $a : c :: b : d$.

Because the quantities are proportionals, $\frac{a}{b} = \frac{c}{d}$;

and multiplying by $\frac{b}{c}$, $\frac{a}{c} = \frac{b}{d}$, or $a : c :: b : d$.

45. Unless the four quantities are of the same kind, the alternation cannot take place, because this operation supposes the first to be some multiple, part or parts, of the third.

One line may have to another line the same ratio that one weight has to another weight, but a line has no relation in respect of magnitude to a weight. In cases of this kind, if the four quantities be represented by numbers or other quantities which are similar, the alternation may take place, and the conclusions drawn from it will be just.

46. If $a : b :: c : d$, then *componendo*,

$$a + b : b :: c + d : d.$$

For $\frac{a}{b} = \frac{c}{b}$; therefore $\frac{a}{b} + 1 = \frac{c}{d} + 1$;

$$\text{therefore } \frac{a+b}{b} = \frac{c+d}{d};$$

therefore $a + b : b :: c + d : d$.

47. Also *dividendo*, $a - b : b :: c - d : d$.

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} - 1 = \frac{c}{d} - 1$;

$$\text{therefore } \frac{a-b}{b} = \frac{c-d}{d};$$

$$\text{therefore } a-b : b :: c-d : d.$$

48. Also *convertendo*, $a : a-b :: c : c-d$.

$$\text{For } \frac{a}{b} = \frac{c}{d}; \text{ therefore } \frac{b}{a} = \frac{d}{c};$$

$$\text{therefore } 1 - \frac{b}{a} = 1 - \frac{d}{c};$$

therefore $\frac{a-b}{a} = \frac{c-d}{c}$; therefore $a-b : a :: c-d : c$;

and by Art. 43, $a : a-b :: c : c-d$.

49. If we have any number of sets of proportionals, and if the corresponding terms be multiplied together, the products are proportionals.

$$\text{If } a : b :: c : d, \text{ and } p : q :: r : s,$$

$$\text{and } u : v :: x : y,$$

$$\text{then } apu : bqv :: crx : dsy.$$

$$\text{For } \frac{a}{b} = \frac{c}{d}, \text{ and } \frac{p}{q} = \frac{r}{s}, \text{ and } \frac{u}{v} = \frac{x}{y};$$

and multiplying together equals $\frac{apu}{bqv} = \frac{crx}{dsy}$;

$$\text{therefore } apu : bqv :: crx : dsy.$$

50. If the same quantities occur in the antecedents of one set of proportionals and the consequents of another set, the resulting proportionals will be reduced.

$$\text{If } a : b :: c : d, \text{ and } b : e :: d : f,$$

$$\text{then } a : e :: c : f.$$

$$\text{For } \frac{a}{b} = \frac{c}{d} \text{ and } \frac{b}{e} = \frac{d}{f}; \text{ therefore } \frac{ab}{be} = \frac{cd}{df}.$$

$$\text{and } \frac{a}{e} = \frac{c}{f}; \text{ wherefore } a : e :: c : f.$$

If $a : b :: x : y$, and $b : c :: z : x$,

and $c : d :: z : t$,

then $a : d :: z^2 : ty$.

$$\text{For } \frac{a}{b} = \frac{x}{y}, \text{ and } \frac{b}{c} = \frac{z}{r}, \text{ and } \frac{c}{d} = \frac{z}{t};$$

$$\text{therefore } \frac{abc}{bcd} = \frac{xzz}{yxt};$$

and expunging common factors in the numerators and denominators

$$\frac{a}{d} = \frac{z^2}{yt}.$$

* (11.) *Of Variation.*

51. Quantities of the same kind assume different values under constant conditions, and when these different values are compared, the quantities are spoken of as *variable*, and the proportion of the different values may be expressed by two terms of a proportion instead of four.

Thus if a man travel with a constant velocity (for example 4 miles an hour,) the space travelled over in any one time is to the space travelled over in any other time as the first time is to the second time; and this may be expressed by saying that the space *varies as* the time, or *is as* the time.

52. One quantity is said to *vary directly* as another when the two quantities depend wholly upon each other, in such a manner that if the one be changed the other is changed in the same proportion.

If the altitude of a triangle be invariable, the area varies as the base. For if the base be increased or diminished in any proportion, the area is increased or diminished in the same proportion. (Euc. VI. 1.)

53. One quantity is said to vary *inversely* as another, when the former cannot be changed in any manner, but the reciprocal of the latter is changed in the same manner.

If the area of a triangle be given the base varies as the perpendicular altitude.

If A, a represent the altitudes, B, b the bases of two triangles, since a triangle is half the rectangle on the same base and of the same altitude, and the triangles are equal, $\frac{1}{2}AB = \frac{1}{2}ab$. (See Geometry.)

Therefore

$$A : a :: b : B, \text{ or } A : a :: \frac{1}{B} : \frac{1}{b}.$$

54. One quantity is said to *vary as others jointly*, if, when the former is changed in any manner, the product of the others is changed in the same proportion.

The area of a triangle varies as its altitude and base jointly.

Let A, B, a, b be the altitudes and bases of two triangles as before, and S, s the areas; then

$$S = \tfrac{1}{2}AB, \; s = \tfrac{1}{2}ab \text{ and } S : s :: AB : ab.$$

55. In the same manner $A : a :: \frac{S}{B} : \frac{s}{b}$; and A varies as S directly and B inversely.

56. The symbol $\propto$ is often used for variation. Thus the above variations may be expressed

$$A \propto \frac{1}{B}, \; S \propto AB, \; A \propto \frac{S}{B}.$$

57. When the increase or decrease of one quantity depends upon the increase or decrease of two others, and it appears that if either of these latter be constant, the first varies as the other, when they both vary, the first varies as their product.

Thus, if V be the velocity of a body moving uniformly, T the time of motion, and S the space described; if T be constant $S \propto V$; if V be constant $S \propto T$; but if neither be constant $S \propto TV$.

Let s, v, t be any other velocity, space and time; and let X be the space described with the velocity v in the time T: then

$S : X :: V : v$, because T is the same in both,

$X : s :: T : t$, because v is the same in both.

Therefore (Art. 50.)

$S : s :: TV : tv$; that is, $S \propto TV$.

(12.) *Of Arithmetical Progression.*

58. Quantities are said to be in arithmetical progression, when they increase or decrease by a common difference.

Thus 1, 3, 5, 7, 9, &c., where the increase is by the difference 2;

a, $a+b$, $a+2b$, $a+3b$, &c. where the increase is by the difference b;

$9a+7x$, $8a+6x$, $7a+5x$, &c. where the decrease is by the difference $a+x$;

are in arithmetical progression.

59. To find any term of an arithmetical progression, multiply the difference by the number of the term *minus* one, and add the product to the first term,

if the progression be an increasing one, or subtract the product, if a decreasing one.

Thus the 10^{th} term of 1, 3, 5, &c. is $1 + 9 \times 2 = 19$.

The n^{th} term of a, $a + b$, $a + 2b$, &c. is $a + \overline{n - 1}\, b$.

The 6^{th} term of $9a + 7x$, $8a + 6x$, &c. is

$9a + 7x - 5(a + x) = 9a + 7x - 5a - 5x = 4a + 2x$.

60. To find the sum of an arithmetical progression, multiply the sum of the first and last terms by half the number of terms.

Thus the sum of 10 terms of 1, 3, 5, &c. is

$$(1 + 19) \times 5 = 100.$$

$$\begin{array}{llr}
\text{For if} & 1 + \;\;3 + \;\;5 + \&c. & \text{to } 19 \text{ (10 terms)} = s, \\
 & 19 + 17 + 15 + \&c. & \text{to } \;\;1 \text{ (10 terms)} = s; \\
\text{therefore} & 20 + 20 + 20 + \&c. & \text{to } 20 \text{ (10 terms)} = 2s,
\end{array}$$

or $20 \times 10 = 2s$, or $20 \times 5 = s$.

Also n terms of a, $a + b$, $a + 2b$, &c.

$$= (2a + \overline{n - 1}\, b)\, \frac{n}{2}.$$

For if $a + (a + b) + (a + 2b) + \&c.$

to $a + \overline{n - 1}\, b$ (n terms) $= s$

$(a + \overline{n - 1}\, b) + (a + \overline{n - 2}\, b) + \&c.$

to a (n terms) $= s$;

therefore $(2a + \overline{n - 1}\, b) + (2a + \overline{n - 1}\, b) + \&c.$

(n terms) $= 2s$;

therefore $(2a + \overline{n - 1}\, b) \times n = 2s$

and $(2a + \overline{n - 1}\, b) \times \dfrac{n}{2} = s$.

(13.) *Of Geometrical Progression.*

61. Quantities are said to be in geometrical progression, or continual proportion, when the first is to the second as the second to the third, and as the third to the fourth, &c.

Or when every succeeding term is a certain multiple or part of the preceding term.

Thus 8, 12, 18, 27 are in continued proportion or in geometric progression. In this case the terms are

$$8, \quad 8 \times \frac{3}{2}, \quad 8 \times \frac{3}{2} \times \frac{3}{2}, \quad 8 \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2},$$

$$\text{or } 8, \quad 8 \times \frac{3}{2}, \quad 8 \times \left(\frac{3}{2}\right)^2, \quad 8 \times \left(\frac{3}{2}\right)^3.$$

In like manner a, ar, ar^2, ar^3 are in geometric progression.

62. The multiplier by which each term is obtained from the preceding is called the *common ratio.*

63. To find any term of a geometrical progression, multiply the first term by that power of the common difference which has for its exponent the number of the term *minus* one.

Thus the 5th term of the progression 8, 12, 18, &c. is,

$$8 \left(\frac{3}{2}\right)^4 = 8 \times \frac{81}{16} = \frac{81}{2} = 40\tfrac{1}{2}.$$

And the n^{th} term of a, ar, ar^2, &c. is ar^{n-1}.

64. To find the sum of an increasing geometrical progression, multiply the last term by the common ratio, subtract from the product the first term, and divide the remainder by the excess of the common ratio above unity.

Thus the sum of 5 terms of 8, 12, 18, &c. is

$$\frac{\frac{81}{2} \times \frac{3}{2} - 8}{\frac{3}{2} - 1} = \frac{243 - 32}{4} \div \frac{1}{2} = \frac{211}{2} = 105\tfrac{1}{2}.$$

And the sum of n terms of a, ar, ar^2, &c. is

$$\frac{ar^{n-1} \times r - a}{r - 1} = \frac{ar^n - a}{r - 1}.$$

For if $a + ar + ar^2 + \&c.$
$+ ar^{n-1}$ (n terms) $= s$,

multiplying by r, $ar + ar^2 + \&c.$
$+ ar^{n-1} + ar^n$ (n terms) $= rs$,

and subtracting, $ar^n - a = rs - s = (r - 1)s$,

whence $\dfrac{ar^n - a}{r - 1} = s$.

GEOMETRY.

ELEMENTS OF GEOMETRY. EUCLID, Books *I, *II, *III, IV.

Book V. **Definition of Proportion.*

The first of four magnitudes is said to have the same ratio to the second which the third has to the fourth, when—any *equi*-multiples *whatsoever* of the *first* and *third* being taken, and any equi-multiples whatsoever of the *second* and *fourth*,—if the multiple of the first be *less* than that of the second, the multiple of the third is also *less* than that of the fourth; or if the multiple of the first be *equal* to the multiple of the second, the multiple of the third is also *equal* to that of the fourth; or if the multiple of the first be *greater*

than that of the second, the multiple of the third is also *greater* than that of the fourth.

Ratio is the relation of quantities in respect of proportion, so that if a, b, c, d be proportional, the ratio of a to b is equal to the ratio of c to d.

*LEMMA 1. If magnitudes be proportionals according to the algebraical definition of proportion, they are also proportionals according to the geometrical definition.

If magnitudes a, b, c, d be proportionals algebraically, $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{ma}{nb} = \frac{mc}{nd}$, where ma, mc are any equi-multiples of a, c, and nb, nd, any equi-multiples of b, d; and if ma be less than nb, mc is less than nd; and if equal, equal; and if greater, greater. (Art. 38.) Therefore the magnitudes a, b, c, d are proportionals according to the geometrical definition.

LEMMA 2. If magnitudes be proportionals according to the geometrical definition, they are also proportionals according to the algebraical definition.

If $a : b :: c : d$ according to the geometrical definition, suppose, first, a to be any multiple, part, or parts of b, so that $a = \frac{n}{m} b$; therefore $ma = nb$, therefore by the definition $mc = nd$; therefore $\frac{c}{d} = \frac{n}{m}$; also $\frac{a}{b} = \frac{n}{m}$; therefore $\frac{a}{b} = \frac{c}{d}$.

Hence $\frac{a}{b} = \frac{c}{d}$, whenever a is any multiple, part, or parts of b. But when the quantities a, b, c, d are determined by any geometrical conditions, the fractions

$\frac{a}{b}$ and $\frac{c}{d}$ will be equal or unequal according to those conditions, and the algebraical equation will express the results of these conditions generally, without regard to magnitude. Therefore the equality cannot depend upon that particular magnitude of a or b, which makes a some multiple, part, or parts of b. Therefore, since, for those magnitudes of a and b for which a is a multiple, part, or parts of b, $\frac{a}{b}$ is equal to $\frac{c}{d}$, these fractions must be equal without any such restriction, and we shall have in all cases $\frac{a}{b} = \frac{c}{d}$.

Hence when quantities have been proved to be geometrically proportional, we may apply to them all those results of algebraical proportion which have been already proved, in Arts. 38 to 50.

EUCLID, Book VI.

Definition I. The altitude of any figure is the straight line drawn from the vertex perpendicular to the base.

Def. 2. Similar rectilineal figures are those which have their several angles respectively equal, and the sides about the equal angles respectively proportionals.

*Prop. I. Triangles and parallelograms of the same altitude are to one another as their bases.

*Prop. II. If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or those produced, proportionally; and if the sides, or the sides produced, be cut propor-

tionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.

*Prop. III. If the angle of a triangle be divided by a straight line cutting the base, the segments of the base shall have the same ratio as the other side of the triangle; and if the segments of the base are to each other as the other sides of the triangle, the straight line drawn from the vertex to the point of section, bisects the vertical angle.

Prop. A. If the exterior angle of a triangle, made by producing one of its sides, be bisected by a straight line, which also cuts the base produced; the segments between the dividing line and the extremities of the base are to each other as the other sides of the triangle; and if the segments of the base produced are to each other as the other sides of the triangle, the straight line drawn from the vertex to the point of section divides the exterior angle of the triangle into two equal angles.

*Prop. IV. The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides; that is, are the antecedents or consequents of the ratios.

Cor. to Prop. IV. Since it has been shewn (Lemma 2) that when quantities are proportionals geometrically, they are proportionals algebraically: all the consequences which are proved of algebraical proportion (Arts. 37 to 50) may be asserted of the proportionals in Props. I, II, III, A, IV of this Book VI.

EUCLID, Book XI.

Def. 1. A straight line is perpendicular or at right angles to a plane, when it makes right angles with every straight line meeting it in that plane.

Def. 2. A plane is parallel to another plane when they do not meet, though both are indefinitely produced.

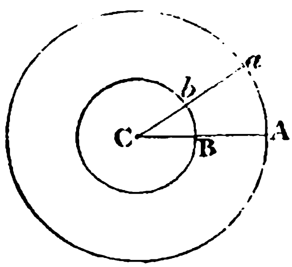
Def. 3. A plane is parallel to a straight line when they do not meet, though both are indefinitely produced.

Def. 4. A *prism* is a solid figure contained by two parallel planes, and by a number of other planes all parallel to one straight line, and cutting the first two planes so as to form polygons.

The first two planes are called the *ends* or *bases* of the prism, and the straight line to which all the other planes are parallel is the *length* of the prism.

The following Lemmas will be taken for granted:

Lemma 3. The arcs which subtend equal angles at the centers of two circles are as the radii of the circles.



Let the two circles be placed so that their centers coincide at C: and so that one of the lines CA containing the angle ACa in one of the circles coincides with the corresponding line CB in the other circle. Then since the angles at the center in the two circles are equal; the other lines containing the angles, namely Ca, Cb, will coincide. And it will be true that $Aa : Bb :: CA : CB$.

Lemma 4. The area of a rectangle is equal to the product of the two sides.

If A, B be the two sides, the rectangle is $= A \times B$.

Cor. If B be the base and A the altitude of a triangle, the area of the triangle is $= \frac{1}{2} A \times B$.

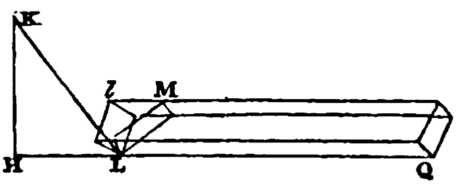
Lemma 5. If a prism be cut by planes perpendicular to its length at different points, the areas of the sections are all similar and equal.

Lemma 6. The solid content of a prism is equal to the product of its length and of the area of a section perpendicular to the length.

If A be the area of the section and H the length, the solid content is $= A \times H$. In this case, solid contents are measured by the number of times they contain a unit of solid content.

Cor. In a uniform prism the weight is as the solid content; hence the weight of any portion of a uniform prism is proportional to its length.

Lemma 7. If a prism be cut by two planes passing through any point of its length, one of the planes being perpendicular to the length and the other oblique to it;



and if a line be drawn at the point, perpendicular to the oblique section and intercepted by a line perpendicular to the length; the oblique section is to the perpendicular section as the portion of the perpendicular line intercepted is to the portion of the length intercepted.

Let Ll, LM be the perpendicular and the oblique section of the prism, of which the length is QL; LK perpendicular to the section LM, and KH perpendicular to the length QL. Then area LM : area Ll :: $KL : HL$.

MECHANICS.

BOOK I. STATICS.

DEFINITIONS AND FUNDAMENTAL NOTIONS.

1. MECHANICS is the science which treats of the laws of the motion and rest of bodies.

2. Any cause which moves or tends to move a body, or which changes or tends to change its motion, is called FORCE.

3. BODY or MATTER is anything extended, and possessing the power of resisting the action of force.

A *rigid* body is one in which the force applied at one part of the body is transferred to another part, the relative positions of the parts of the body not being capable of any change.

4. All bodies within our observation fall or tend to fall to the earth: and the force which they exert in consequence of this tendency, is called their WEIGHT.

5. Forces may produce either rest or motion in bodies. When forces produce rest, they *balance* each other; they are in *equilibrium*; they *destroy* each other's effects.

6. STATICS is the science which treats of the laws of forces in equilibrium.

7. Two directly opposite forces which balance each other are *equal*.

Forces are directly opposite when they act in the same straight line in opposite directions.

8. Forces are capable of *addition*. Thus, when two men pull at a string in the same direction, their forces are added; and when two heavy bodies are put in the same vessel suspended by a string, their weights are added, and are supported by the string.

9. A force is *twice* as great as a given force, when it is the sum of two others, each equal to the given force; a force is *three* times as great, when it is the sum of three such forces; and so on.

10. Forces (in Statics) may be *measured* by the weights which they would support.

11. The *Quantities of Matter* of bodies are measured by the proportion of their mechanical effect.

12. The quantities of matter of two bodies are *as their weights* at the same place.

13. The *Density* of a body is measured by the quantity of matter contained in a given space.

14. A LEVER is a rigid rod, moveable, in one plane, about a point, which is called the *fulcrum* or *center of motion*, by means of forces which tend to turn it round the fulcrum.

15. The portions of the rod between the fulcrum and the points where the forces are applied, are called the *arms*.

16. When the arms are two portions of the same straight line, the lever is called a *straight* lever; otherwise it is called a *bent* lever.

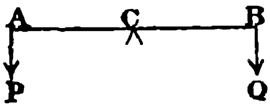
17. The lever is supposed to be without weight, unless the contrary be expressed.

AXIOMS.

1. In a system which is in equilibrium, there is at every point a reaction equal and opposite to the action.

2. If two equal forces act perpendicularly at the extremities of equal arms of a straight lever to turn it opposite ways, they will keep each other in equilibrium.

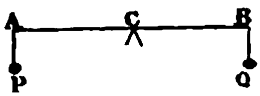
If $AC = BC$, and P and Q be two equal forces acting perpendicularly on CA and CB at A and B, they will balance each other.



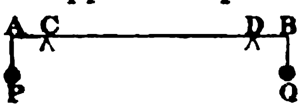
3. If forces keep each other in equilibrium, and if any force be added to one of them, it will preponderate.

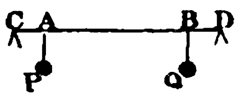
4. If two equal weights balance each other upon a horizontal straight lever, the pressure upon the fulcrum is equal to the sum of the weights, whatever be the length of the lever.

If P, Q be two equal weights which balance each other upon the horizontal lever AB, the pressure upon C is $P + Q$.



5. If two equal weights be supported upon a straight lever on two fulcrums at equal distances from the weights, the pressures upon the two fulcrums are together equal to the sum of the weights.



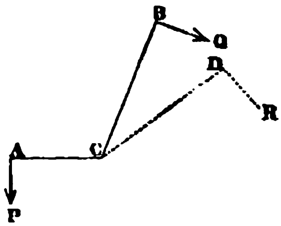


If P, Q be two equal weights which are supported upon the line AB on two fulcrums C, D, so that AC, BD are equal; the pressures upon C, D are together equal to the sum of the weights $P + Q$.

6. On the same suppositions, the pressures on the two fulcrums are equal.

7. If a force act perpendicularly on the straight arm of a bent lever at its extremity, the effect to turn the lever round the fulcrum will be the same, whatever be the angle which the arm makes with the other arm, so long as the length is the same.

If a force Q act perpendicularly on CB at its extremity B, C being the fulcrum, and an equal force R act perpendicularly on an equal arm CD, at its extremity, the effect to turn the lever round C in the two cases is equal.



8. When a force acts upon a rigid body it will produce the same effect to urge the body in the line of its own direction, at whatever point of the direction it acts.

9. If a body which is moveable about an axis be acted upon by two equal forces, in two planes perpendicular to the axis, the forces being perpendicular at the extremities of two straight arms of equal length from the axis; the two forces will produce equal effects to turn the body, at whatever points the arms meet the axis.

10. If a string pass freely round a fixed body, so that the direction of the string is altered, any force exerted at one extremity of the string will produce at the other extremity the same effect as if the force had acted directly.

11. If in a system which is in equilibrium, there be substituted for the force acting at any point, an immoveable fulcrum at that point, the equilibrium will not be disturbed.

12. If in a system which is in equilibrium there be substituted for an immoveable point or fulcrum the force which the fulcrum exerts, the equilibrium will not be disturbed.

13. A perfectly hard and smooth surface, acted on at any point by any force, exerts a reaction which is perpendicular to the surface at that point; and if the surface be supposed to be immoveable, the force will be supported, whatever be its magnitude.

14. A heavy material straight line, prism, or cylinder, of uniform density, may be supposed to be composed of a row of heavy points of equal weight, uniformly distributed along the line.

15. A heavy material plane of uniform density may be supposed to be composed of a collection of parallel straight lines of equal density, uniformly distributed along the plane.

16. A heavy solid body of uniform density may be supposed to be composed of a collection of particles, the weight of each of which is as the portion of the body which it occupies, and which may be considered as heavy points.

POSTULATES.

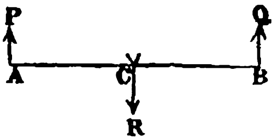
1. A prism or cylinder of uniform density, and of given length, may be taken, which is equal to any given weight.

2. A force may be taken equal to the excess of a greater given force over a less.

3. A force may be taken in a given ratio to a given force.

Proposition I. If a weight be supported on a horizontal rod by two forces acting vertically at equal distances from the weight, the forces are equal to each other, and their sum is equal to the weight.

Let the two forces P, Q act perpendicularly at the extremities of the equal arms CA, CB of the horizontal lever AB; and let them balance each other. The forces P, Q, will be equal;



for if not, let one of them, as P, be the less, and by Post. 2 take X, the force which is the excess of Q above P, so that $P+X$ is equal to Q; therefore, by Ax. 2, $P+X$ will balance Q. But since P balances Q, if we add to P the force X it will preponderate, by Ax. 3; which is absurd. Therefore P is not less than Q; and in the same manner it may be shewn that Q is not less than P. Therefore P and Q are equal.

Hence, since P and Q are equal, by Ax. 5, the pressure on the fulcrum C is equal to the sum of the two forces P, Q. Hence, by Ax. 12, if, instead of a fulcrum, there be a force R, acting at C perpendicularly to the lever, and equal to the sum of P and Q, this force will balance the pressure at C, just as the fulcrum does, and there will be an equilibrium; that is, a vertical force or weight R will be supported by two forces P, Q, acting vertically at equal distances CA, CB; and these forces will be equal; and the weight R is equal to the sum of P and Q. Q.E.D.

* **Prop. II.** A horizontal prism or cylinder of uniform density will produce the same effect by its weight as if it were collected at its middle point.

Let AB be the prism or cylinder, and C its middle point. Let P, R be any points in AC, and let CQ be taken equal to CP, and CS equal to CR.

The half AC of the prism may (by Ax. 14) be supposed to be made up of small equal weights, distributed along the whole of the line AC, as at P, R; and the half BC may in like manner be conceived to be made up of small equal weights distributed along BC; as at Q, S; of which the weight at Q is equal to the weight at P, that at S to that at R, and so on.

Let F be a fulcrum about which the prism AB tends to turn by its weight. In CB, produced if necessary, take CG equal to CF, and suppose a fulcrum placed at G.

Let the weights at P, Q, R, S be denoted by P, Q, R, S.

The two weights P and Q produce upon the fulcrums F and G pressures which together are equal to the sum of the weights $P+Q$, (Ax. 5,) or to the double of P, since P and Q are equal. But the pressure upon each of these fulcrums is equal, (Ax. 6,) hence the pressure upon each of them is P; therefore the pressure upon the fulcrum G, arising from the two weights P and Q, is P; in like manner the pressure upon the fulcrum G, arising from R and S, is R; and so of the rest: and the whole pressure on G, arising from the whole prism AB, is the sum of all the weights P, R &c. from A to C; that is, it is half the weight of the prism.

But if the whole prism be collected in its middle point C, the pressure upon the two fulcrums F and

G will be the whole weight of the prism, and the pressures on the two fulcrums are equal, by Prop. 1. Therefore, in this case also, the pressure on the fulcrum G is equal to half the weight of the prism. Therefore the prism, when collected at its middle point, produces the same pressure on the fulcrum G as it did before.

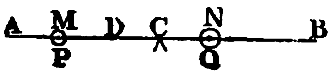
Therefore, when a uniform prism is collected at its middle point, it produces the same effect by its weight as it did before. Q. E. D.

COR. 1. A uniform prism or cylinder will balance itself upon its middle point.

COR. 2. When a prism or cylinder thus balances upon its middle point, the pressure upon the fulcrum is equal to the weight of the prism.

* PROP. III. **If two weights, acting perpendicularly on a straight lever on opposite sides of the fulcrum, are inversely as their distances from the fulcrum, they will balance each other; and the pressure on the fulcrum will be equal to their sum.**

Let P, Q be the two weights, MCN the lever. Let NC be the less of the two NC, CM. Take MD and MA each equal to NC, and NB equal to ND. Let there be a uniform prism of the length AB, equal in weight to $P+Q$ (Post. 1). Since MD is equal to CN, adding CD to both, MC is equal to DN. Therefore AD, which is double of MD, is double of CN; and BD which is double of DN, is double of CM. Therefore $AD : BD :: CN : CM$; and by supposition, $CN : CM :: P : Q$; therefore $AD : BD :: P : Q$, and componendo $AD + BD : AD :: P + Q : P$. But



$AD+BD$ or AB is equal in weight to $P+Q$; and the prism AB is uniform; therefore, by Cor. to Lemma 6, AD is equal in weight to P: in like manner BD is equal in weight to Q.

Now since AM is equal to CN, and MC to NB, the sum AC is equal to the sum CB; and the prism AB will balance upon its middle point C. (Prop. 2; Cor. 1.)

But by Prop. 2. if the prism AD be collected at its middle point M, and the prism BD at its middle point N, the effect will be the same as before; therefore, in this case also, the weights will balance upon C; that is, the weight P at M, and Q at N, will balance upon C.

Therefore if the weights P, Q be inversely as their distances CM, CN, they will balance each other. Q. E. D.

Also the pressure of the prism AB upon the fulcrum C is equal to the weight of the prism, that is, to the weight $P+Q$. (Prop. 2, Cor. 2.) And by Prop. 2, when AD is collected at M and BD at N, the pressure on C is not altered; that is, when P is at M and Q at N, the pressure upon C is $P+Q$, the sum of the weights. Q. E. D.

* Prop. IV. **If two weights acting perpendicularly on a straight lever on opposite sides of the fulcrum balance each other, they are inversely as their distances from the fulcrum.**

Let P, Q, be two weights which balance each other on the lever MCN; then $NC : CM :: P : Q$.

If not, let $NC : CM :: P : Y$, and first, let Y be less than Q; so that $Q = Y + E$. By Prop. 3,

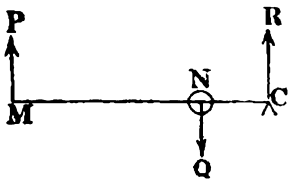
P and Y will balance each other. And therefore when E is added to Y, by Axiom 4, $Y + E$ or Q will preponderate against P; but by hypothesis, P and Q balance, which is absurd: therefore Y is not less than Q.

Nor is it greater; for if so, let $NC : CM :: X : Q$, and since $NC : CM :: P : Y$, we have $P : Y :: X : Q$; and since Y is greater than Q, P is greater than X: let $P = X + D$. Then, since $NC : CM :: X : Q$, by Prop. 3, X and Q will balance each other. And therefore when D is added to X, by Axiom 3, $X + D$ or P will preponderate against Q; but by hypothesis, P and Q balance, which is absurd; therefore Y is not greater than Q.

Therefore we cannot have $NC : CM :: P : Y$, Y being a quantity less or greater than Q. Therefore, if P and Q balance, $NC : CM :: P : Q$. Q.E.D.

* Prop. V. If two forces, acting perpendicularly on a straight lever in opposite directions and on the same side of the fulcrum, are inversely as their distances from the fulcrum, they will balance each other, and the pressure on the fulcrum will be equal to the difference of the forces.

Let MCN be the lever on which two forces P, Q balance each other, the fulcrum being at C. Let MNC be supposed to be a lever on which two forces P, R, acting perpendicularly at M, C, on opposite sides of the fulcrum N, balance each other.



Then by Prop. 4, $R : P :: MN : NC$; and therefore $R + P : P :: MN + NC : NC$, that is, $R + P : P :: MC : NC$. But (by Prop. 3) the pressure upon the fulcrum N is equal to $R + P$, and is in a direction opposite to the

forces P and R. If therefore a force Q equal to $R+P$, act perpendicularly to the lever MC at N in the direction opposite to P and R, it will supply the place of the fulcrum, and the forces will still balance each other, by Ax. 12. But if we place an immoveable fulcrum at C, it will supply the place of the force R, and the forces P, Q will still balance each other, by Ax. 11. That is, if $Q : P :: MC : NC$, the forces P, Q will balance each other on the lever MCN, of which the fulcrum is at C.

Also the pressure on the fulcrum C is equal to the force R, which is the difference of $P+R$ and P, that is, of Q and P. Q. E. D.

Cor. In nearly the same manner, by means of Prop. 4, we may prove the converse proposition; that if two forces, acting perpendicularly on a straight lever, in opposite directions and on the same side of the fulcrum, balance each other, they are inversely as their distances from the fulcrum.

* Prop. VI. To explain the different kinds of levers.

When material levers are used, the two forces which have been spoken of, as balancing each other upon the lever, are exemplified by the weight to be raised or the resistance to be overcome, as the one force, and the pressure, weight, or force of any kind, employed for the purpose, as the other force. The former of these forces is called *the Weight*, the latter is called *the Power*.

The preceding Propositions give the proportion of the Power and Weight in the case of equilibrium, that is, when the weight is not raised, but only supported; or when the resistance is not overcome, but

only neutralized. But knowing the Power which will produce equilibrium with the Weight, we know that any additional force will make the Power preponderate. (Ax. 3.)

Straight levers are divided into three kinds, according to the position of the Power and Weight.

1. The Lever of the First kind is that in which the Power and Weight are on opposite sides of the Fulcrum, as in Propositions 3 and 4.

We have an example of a lever of this kind, when a bar is used to raise a heavy stone by pressing down one end of the bar with the hand, so as to raise the stone with the other end: the Power is the force of the hand, the Fulcrum is the obstacle on which the bar rests, the Weight is the weight of the stone.

We have an example of a double lever of this kind in a pair of pincers used for holding or cutting; the Power is the force of the hand or hands at the handle, the Weight is the resistance overcome by the pinching edges of the instrument, the Fulcrum is the pin on which the two pieces of the instrument move.

2. The Lever of the Second kind is that in which the Power and the Weight are on the same side of the Fulcrum, the Weight being the nearer to the Fulcrum.

We have an example of a lever of this kind, when a bar is used to raise a heavy stone by raising one end of the bar with the hand, while the other end rests on the ground, and the stone is raised by an intermediate part of the bar. The Fulcrum is the ground, the Power is the force exerted by the hand, the Weight is the weight of the stone.

We have an example of a double lever of this kind in a pair of nutcrackers. The Power is the force of the hand exerted at the handles; the Weight

is the force with which the nut resists crushing; the Fulcrum is the pin which connects the two pieces of the instrument.

3. The Lever of the Third kind is that in which the Power and the Weight are on the same side of the fulcrum, and the Weight is the further from the fulcrum.

In this kind of lever, the Power must be greater than the weight in order to produce equilibrium, by Prop. 5. Therefore by the use of such a lever, force is lost. The advantage gained by the lever is, that the force exerted produces its effect at an increased distance from the fulcrum.

We have an example of a lever of this kind in the anatomy of the fore-arm of a man, when he raises a load with it, turning at the elbow. The elbow is the Fulcrum, the Power is the force of the muscle which, coming from the upper arm is inserted into the fore-arm near the elbow, the Weight is the load raised.

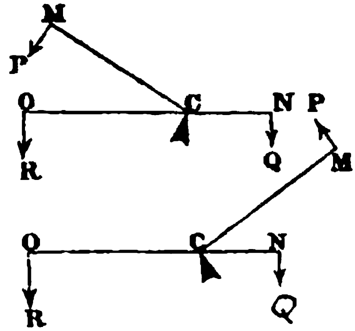
We have an example of a double lever of this kind in a pair of tongs used to hold a coal. The Fulcrum is the pin on which the two parts of the instrument turn, the Power is the force of the fingers, the Weight is the pressure exerted by the coal upon the ends of the tongs.

* PROP. VII. If two forces acting perpendicularly at the extremities of the straight arms of a bent lever are inversely as the arms, they will balance each other.

Let MCN be any lever: let P, Q act perpendicularly on the arms CM, CN, and let $P : Q :: CN : CM$; P and Q will balance.

Produce NC to O, taking CO equal to CM; and at O let a force R equal to P act perpendicularly on the

lever NCO, to turn it in the same direction as P. Then since $P : Q :: CN : CM$, and that R is equal to P, and CO to CM, we shall have $R : Q :: CN : CO$. Therefore, by Prop. 3, the force R will balance the force Q.



But since CM is equal to CO, and the force P to the force R, both acting perpendicularly to the arms, by Axiom 7, P and R will produce the same effect to turn the lever round the fulcrum C; and therefore since R balances Q, P also will balance Q. Q. E. D.

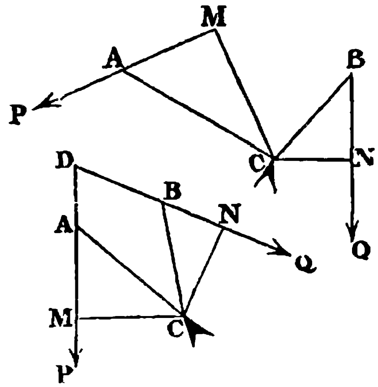
Cor. Conversely, if the forces P and Q balance each other, they are inversely as the arms. For, making the same construction, the force R produces the same effect as the force P; therefore R balances Q. And hence, by Prop. 4, $NC : CO :: Q : R$, that is, $NC : CM :: Q : P$.

* Prop. VIII. If two forces, acting at any angles on the arms of any lever, are inversely as the perpendiculars drawn from the fulcrum to the directions in which the forces act, they will balance each other.

Let P, Q be two forces, acting at any angles on AC, BC, the arms of the lever ACB; and let CM, CN be perpendiculars on the directions AP, BQ (produced if necessary) in which the forces act: if $CM : CN :: Q : P$, the forces P, Q will balance each other.

The lever ACB is supposed to be rigid, so that AC, BC cannot alter their relative position. Hence we may suppose the plane ACB to be a rigid in-

definite plane, moveable about the point C, and AC, BC to be lines in this plane. Therefore the forces P, Q, which act at the points A, B, will, by Axiom 8, produce the same effect as if they act at the points M, N respectively. But if they act at these points they will balance each other, by Prop. 7. Therefore the forces P, Q acting at the points A, B will balance each other. Q.E.D.



Cor. 1. In the same manner it may be proved by means of the Corollary to Prop. 7, that if the forces P, Q balance each other, then will $P : Q :: CN : CM$.

Cor. 2. If P, Q, CM, CN, be expressed in numbers, $P \times CM = Q \times CN$.

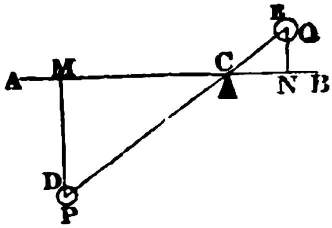
Cor. 3. If X be any force acting on the lever ACB, and CO the perpendicular upon its direction, and if $X \times CO = P \times CM$, the force X will produce upon the lever the same effect as P. For $X \times CO = Q \times CN$; therefore X will balance Q; which is what P does.

Cor. 4. If the two forces P, Q act at the same point D, the proposition is still true.

* Prop. IX. If two weights balance each other on a straight lever when it is horizontal, they will balance each other in every position of the lever.

Let it be supposed that the weights P, Q, acting at A, B, balance each other upon the lever when it is in the horizontal position ACB; the weights P, Q will

balance each other upon the same lever in any other position, as DCE.



Draw DM, CN vertical, meeting the horizontal line ACB. Then, in the triangles DCM, ECN, the vertical angles DCM, ECN are equal; and DMC, ENC are equal, being right angles; therefore the remaining angles of the triangles are equal, and the triangles are equiangular and similar. Therefore $DC : CM :: EC : CN$, and alternately $DC : EC :: CM : CN$. But since P, Q balance each other on AB, $Q : P :: AC : CB$; and AC is equal to DC, and EC to BC, because ACB and DCE are the same lever; therefore $Q : P :: DC : EC$; therefore by what precedes, $Q : P :: CM : CN$; therefore, by Prop. 8, the weights P, Q acting at the points D, E will balance each other. Q. E. D.

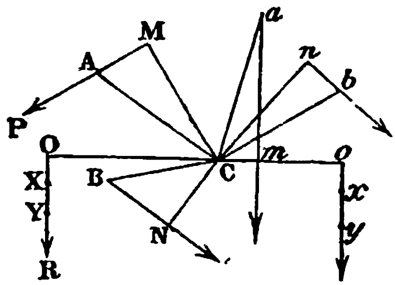
Cor. The pressure upon the fulcrum C in every position of the lever DE is equal to the sum of the weights P and Q. For in every position the effect of the weights P, Q is the same as if they acted at M, N, by Axiom 8. But in this case, by Prop. 3, the pressure on the fulcrum C is the sum of the weights.

Def. If lines be expressed by numbers, the product of a force acting on a lever, by the perpendicular drawn from the axis of motion upon its direction, is called its *moment*.

Prop. X. If any number of forces act upon a lever, and tend to turn it opposite ways, and if the sum of the moments of the forces which tend to turn the lever one way be equal to the sum of the moments of the forces which tend to

turn it the other way, the forces will balance each other.

Let the forces P, Q, R, tend to turn the lever one way, and let CM, CN, CO be the perpendiculars on their directions; and let the forces p, q, tend to turn the lever the other way, and let Cm, Cn be the perpendiculars on their directions; and let $P \times CM + Q \times CN + R \times CO$ be equal to $p \times Cm + q \times Cn$; the forces will balance each other.



Let any two lines CO, Co be taken, and let forces act at O and o, perpendicularly to CO, Co, to turn the lever opposite ways, namely, at O, a force X, such that $CO : CM :: PX$, by Post. 2, that is, such that $X \times CO = P \times CM$; and also a force Y, such that $Y \times CO = Q \times CN$, and a force R; and also at o, a force x, such that $x \times Co = p \times Cm$, and a force y, such that $y \times Co = q \times Cm$.

Then, by Cor. 3 to Prop. 8, the force X will produce the same effect as the force P, and the force Y will produce the same effect as the force Q; and therefore the forces P, Q, R will produce the same effect as X, Y, R acting at O. In like manner the forces p, q will produce the same effect as x, y, acting at o.

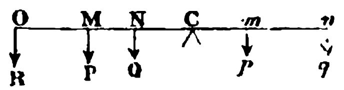
But the forces X, Y, R, acting at O, will balance the forces x, y, acting at o, if $(X + Y + R) \times CO$ be equal to $(x + y) \times Co$, by Prop. 8; that is, if $X \times CO + Y \times CO + R \times CO$ be equal to $x \times Co + y \times Co$; that is, by the construction, if $P \times CM + Q \times CN + R \times CO$ be equal to $p \times Cm + q \times Cn$. Therefore, &c. Q. E. D.

Cor. 1. If the forces be weights acting on a straight horizontal lever, the same is true, putting for the perpendiculars on the directions of the forces, the portions of the lever CM, CN, &c. intercepted between the fulcrum and the weights. (See next figure).

Cor. 2. The converse of this Proposition and of Cor. 1. are true.

Prop. XI. If any number of forces acting perpendicularly upon a lever balance each other, they may be separated, so that, retaining their positions, they form pairs, each of which pairs would balance on the fulcrum separately.

Let P, Q, R, p, q be any forces which balance each other on the lever $OMNCmn$. If each force on one side of the fulcrum has its moment equal to that of a corresponding force on the other side, it is clear that each force will balance the corresponding one on the other side, and the forces are already in such pairs as are mentioned in the Proposition. But if not, let any moment on one side, as $P \times CM$, be less than a moment on the other side, as $p \times Cm$. Assume u such that $Cm : CM :: P : u$, by Post. 2: therefore $P \times CM = u \times Cm$; therefore $u \times Cm$ is less than $p \times Cm$, and u is less than p; let $p = u + x$. Then if p be separated into u and x, the pair P and u will balance each other separately, because their moments are equal.



In the same manner, of the forces Q, R, x, q, take any other as Q, of which the moment $Q \times CN$ is less than the moment of $q \times Cn$ of a force q on the other side of the fulcrum. Assume v such, that $Cn : CN :: Q : v$, therefore $Q \times CN = v \times Cn$; and

let $q = v + y$. Then if q be separated into v and y, the pair Q and v will balance each other separately, for the same reason as before.

And of the forces R, x, y, the moment $x \times Cm$ must be less than $R \times CO$. Assume $X \times CO = x \times Cm$; and let $R = X + Y$. The pair X, x will balance each other separately, as before.

But because the forces P, Q, R, p, q balance on the lever, it follows (by Cor. 2 to Prop. 10) that

$$P \times CM + Q \times CN + R \times CO = p \times Cm + q \times Cn:$$

and hence, since

$$R = X + Y, \text{ and } p = u + x, \text{ and } q = v + y,$$

$$P \times CM + Q \times CN + X \times CO + Y \times CO$$

$$= u \times Cm + x \times Cm + v \times Cn + y \times Cn;$$

and it has been supposed that

$$P \times CM = u \times Cm, \text{ and } Q \times CN = v \times Cn,$$

$$\text{and } X \times CO = x \times Cm;$$

hence the remainder

$$Y \times CO \text{ is } = y \times Cn;$$

and the pairs Y, y will balance each other.

Therefore the forces have been separated into pairs,

$$P, u; \ Q, v; \ X, x; \ Y, y;$$

which balance each other separately. Q. E. D.

Also it is plain that the same proof may be applied in any case; for at each step the number of forces which are not in pairs is diminished by one; and therefore the reduction may always be effected by as many steps as there are forces, wanting one.

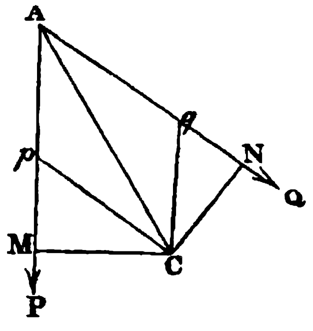
Cor. If any forces act perpendicularly upon a lever, the pressure upon the fulcrum is equal to the sum of the forces. For the pressure upon the fulcrum

arising from each pair is equal to the sum of the two forces of that pair; therefore the whole pressure is equal to the sum of all the pairs, that is, to the sum of all the forces.

* DEF. When two forces act at the same point, they produce the same statical effect as a certain single force, acting at that point. This single force is called the *resultant* of the two; they are called its *components*. The two forces produce the single force by being *compounded*, and it may be *resolved* into the two.

Straight lines may *represent* forces in direction and magnitude when they are taken in the direction of the forces and proportional to their magnitude. When forces are so represented, if AB represent any force, BA represents an equal and opposite force. A force represented by any line, as AB, is often called the force AB.

*** PROP. XII. If the adjacent sides of a parallelogram represent the component forces in direction and magnitude, the diagonal will represent the resultant force in *direction*.**



Let Ap, Aq represent in magnitude and direction the forces P, Q, acting at A; complete the parallelogram $ApCq$; and draw AC; draw also CM, CN perpendicular upon Ap, Aq.

The triangles CpM, CqN have right angles at M and N, and the angles MpC, CqN equal, each being equal to MAN; therefore the triangles

CpM, CqN are equiangular and similar. Therefore $CM : CN :: Cp : Cq$; that is, $CM : CN :: Aq : Ap$. But Ap, Aq represent the forces P, Q in magnitude; therefore $CM : CN :: Q : P$. Therefore, by Prop. 8, if the forces P, Q act on the plane PAQ, supposed to be moveable about the point C, they will balance each other, producing a pressure on the fulcrum C.

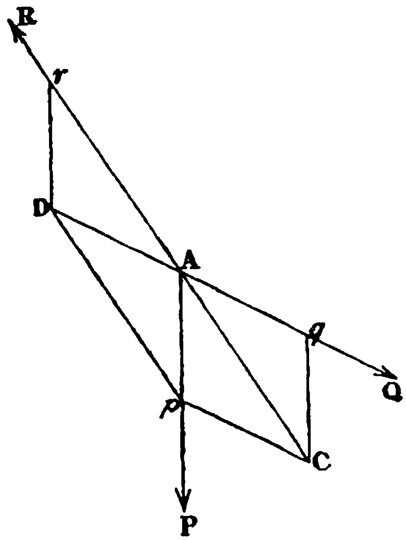
Therefore the single force which produces the same effect as P, Q will produce a pressure upon the point C, but will not turn the plane about C. But this cannot be the case except the single force act in the line AC; for if it acted in any other direction, a perpendicular might be drawn from C upon the direction, and the force would produce motion, by Axiom 3. Therefore the resultant acts in the direction AC. Q.E.D.

Cor. 1. If a point, acted upon by two forces Ap, Aq, be kept at rest by a third force, this force must act in the direction CA. For otherwise it would not balance the force in the direction AC, to which the forces Ap, Aq are equivalent.

Cor. 2. Hence if three forces act on a point, and keep each other in equilibrium, each of them is in the direction of the diagonal of the parallelogram whose sides represent the other two.

* Prop. XIII. If the adjacent sides of a parallelogram represent the component forces in direction and magnitude, the diagonal will represent the resultant force in *magnitude*.

Let Ap, Aq represent the component forces in direction and magnitude. Complete the parallelogram $ApCq$; then by Prop. 12, Cor. 1, the two forces Ap, Aq will be kept in equilibrium by a force in the direction CA. Let Ar represent this force in magnitude. Therefore the three forces Ap, Aq, Ar keep each other in equilibrium. Complete the parallelogram $ApDr$, and draw its diagonal DA. By Prop. 12, Cor. 2, the force Aq is in the direction DA, and therefore DAq is a straight line.



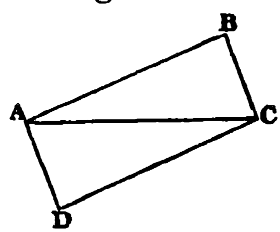
Hence in the triangles CAq, DAr, the vertical angles CAq, DAr are equal, and Cq, Dr are parallel to each other, because Cq and Dr are both parallel to Ap, and Cr meets them, therefore the angle qCA is equal to the alternate angle DrA. Therefore the triangles CAq, DAr are equiangular. Also Cq and Dr are equal, for each is equal to Ap, being opposite sides of parallelograms pq, pr. Therefore (Euc. VI. 8) the other sides of the triangles CAq, DAr are equal; therefore CA is equal to Ar. But Ar represents in magnitude the force which keeps in equilibrium Ap, Aq; and since Ar acting in the opposite direction would balance Ar, the force which produces the same effect as Ap, Aq, is Ar acting in the opposite direction. Therefore AC, which is equal to Ar, represents in magnitude the force which produces the same effect as Ap, Aq; that is, the resultant of Ap, Aq. Q.E.D.

Cor. If the components be represented in magnitude and direction by the sides of a parallelogram, the resultant is represented in magnitude and direction by the diagonal of the parallelogram.

Def. Forces may be *represented* by lines parallel to them in direction and proportional to them in magnitude.

*** Prop. XIV. If three forces, represented in magnitude and direction by the sides of a triangle taken in order, act on a point, they will keep it in equilibrium.**

Let three forces, represented in magnitude and direction by the three lines AB, BC, CA, act on the point A, they will keep it in equilibrium. Complete the parallelogram $ABCD$, then the force which is represented by BC is also represented by AD, and acts at the point A.



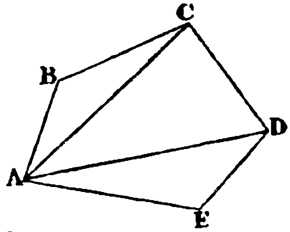
And the resultant of the forces AB, AD is represented in magnitude and direction by AC (Prop. 13, Cor.); therefore the forces AB, BC produce the same effect as AC; and therefore the forces AB, BC, CA produce the same effect as AC, CA; that is, they will keep the point A in equilibrium.

Cor. 1. If three forces which keep a point in equilibrium be in the direction of three lines forming a triangle, they are proportional to those lines.

Cor. 2. Any two forces AB, BC, which act at a point A, are equivalent to a force AC.

Prop. XV. If any number of forces, represented in magnitude and direction by the sides of a polygon taken in order, act on a point, they will keep it in equilibrium.

Let forces AB, BC, CD, DE, EA act upon a point A; they will keep it in equilibrium. By Prop. 14, Cor. 2, the forces AB, BC are equivalent to a force AC; therefore the forces AB, BC, CD are equivalent to the forces AC, CD; that is, by the same corollary, to a force AD. Therefore again, the forces AB, BC, CD, DE are equivalent to the forces AD, DE; that is, again by the same corollary, to a force AE. Therefore, finally, the forces AB, BE, CD, DE, EA are equivalent to forces AE, EA, and therefore will keep the point A in equilibrium.

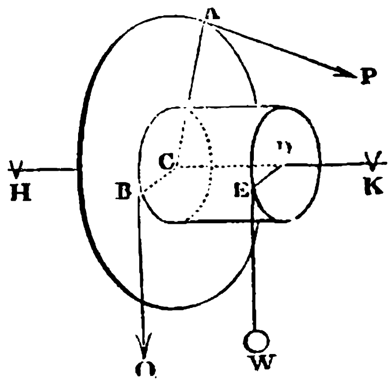


* Def. *The Wheel and Axle* is a rigid machine, which is moveable about an axis, and on which two forces, tending to turn it opposite ways, act in two planes perpendicular to the axis; the one force (the *Power*) acting by means of a string wrapt on the circumference of a circle perpendicular to the axis, called the *Wheel;* the other force (the *Weight*) acting by means of a string wrapt on the surface of a cylinder having the axis of motion for its axis, and called the *Axle.*

* Prop. XVI. There is an equilibrium upon the wheel and axle, when the power is to the weight as the radius of the axle to the radius of the wheel.

Let AB be the wheel, and DEB the axle, the whole being moveable about the axis $HCDK$; the power P, acting at A, perpendicular to CA, the radius of the wheel; and the weight W, acting at B, perpendicular to DE, the radius of the axle. Also let $P : W :: DE : CA$; then there will be an equilibrium.

In the plane of the wheel AB, let CB be drawn from the axis, equal to DE the radius of the axle; and let a force Q, equal to W, act at B perpendicular to CB. Then, by Axiom 9, the two forces Q, W produce equal effects in turning the machine. But the force Q will balance P, by Prop. 7, because $P : W :: DE : CA$, and therefore $P : Q :: CB : CA$, Q being equal to W, and CB to DE: therefore W will balance P, and there will be an equilibrium. Q.E.D.



COR. 1. On the wheel and axle when there is equilibrium, the moments of the power and weight are equal.

COR. 2. If the power and weight do not act perpendicularly to the radii of the wheel and axle, it will appear, by the reasoning of Prop. 8, that there will still be an equilibrium if their moments are equal.

COR. 3. If several forces acting upon a body moveable about a fixed axis, and acting in planes perpendicular to the axis, tend to turn it opposite ways, there will be an equilibrium when the sum of the moments of the forces which tend to turn the body one way is equal to the sum of the moments of the forces which tend to turn the body the other way. This may be proved by reasoning similar to that of Prop. 10.

COR. 4. If a heavy body be moveable about a horizontal axis, it will be in equilibrium when the moments of the weights of the two parts into which it is divided by a vertical plane passing through the

axis, are equal: for these two parts will tend to turn it opposite ways, by Axiom 8.

In this case, the moment of each particle of the body is known by drawing from the particle a vertical line meeting the horizontal plane which passes through the axis. The perpendicular drawn to the axis from the point where the vertical line meets the horizontal plane, multiplied into the weight of the particle, is the moment of the particle.

Cor. 5. Conversely, if these moments are not equal, there cannot be equilibrium.

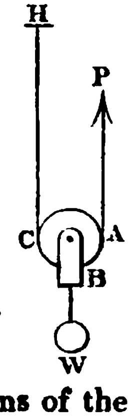
* Def. A *Pully* is a machine in which one part, (the *Block*) being stationary, a string can pass freely round another part, (the *Sheave*).

A pully is fixed when the block is fixed, and moveable when the block is moveable.

The Power is the force which acts at the string; the Weight, is the weight supported.

* Prop. XVII. In the single moveable pully, where the strings are parallel, there is an equilibrium when the power is to the weight as 1 to 2.

Let ABC represent a pully in which B is the block, AC the sheave, and in which the strings PA, HC are parallel: there is an equilibrium when $P : W :: 1 : 2$.



By Axiom 10, since the string passes freely round the sheave AC, the force P, which is exerted on the string PA, is equal to that which the string CH exerts on the fixed point H; and therefore the reaction which the fixed point H exerts by means of the string HC, is also equal to P. And the two forces, each equal to P, which act by means of the

parallel strings AP, CH, may be considered as balancing each other upon a lever AC, the fulcrum of which is in the point of the block B, by which the weight W is supported. Therefore, by Prop. 3, the pressure on the fulcrum is the sum of these forces, that is, it is the double of P; and this pressure on the fulcrum of the block B is balanced by the pressure or weight of W upon the block in the opposite direction, in the case of equilibrium: therefore, in the case of equilibrium, W is double of P, or $P : W :: 1 : 2$.

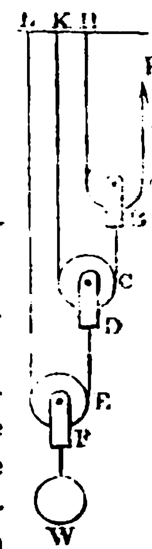
* Prop. XVIII. In a system in which the same string passes round any number of pulleys, and the parts of it between the pulleys are parallel, there is an equilibrium when power (P) : weight (W) :: 1 : the number of strings at the lower block.

Let AC represent the system of pulleys; the string $ABCDEFGHK$ passing round all the pulleys, and the portions CB, DE, GF, HK, being all parallel. By Axiom 10, the forces exerted by all these strings will be equal to P; therefore the forces which they exert upon the lower block will each be equal to P. And these forces may be considered as acting upon a lever, the fulcrum of which is in the point of the block Z, by which the weight W is supported. Therefore by Prop. 11, Cor. the pressure upon this fulcrum is equal to the sum of the forces of the strings, that is, it is as many times P as there are strings at the lower block. And this pressure on the fulcrum in the lower block is balanced by the pressure or weight of W in the

opposite direction in the case of equilibrium; therefore in the case of equilibrium, $P : W :: 1 :$ number of strings in the lower block.

* **PROP. XIX.** In a system in which each pully hangs by a separate string, and the strings are parallel, there is an equilibrium when $P : W :: 1 :$ that power of 2 whose index is the number of moveable pulleys.

Let AL represent the system of pulleys; each pully A, C, E hanging by a separate string, and the strings being all parallel. It appears by the reasoning of Prop. 17, that



P : force of string $BC :: 1 : 2$;

force of string BC : force of string $DE :: 1 : 2$;

force of string DE : force of string $FW :: 1 : 2$.

And there will be as many such proportions as there are moveable pulleys A, C, E. Also in compounding these proportions, the proportion compounded of the former ratios in each proportion will be P : force of string FW; and the proportion compounded of the latter ratios in each proportion will be $1 : 2$ raised to that power whose index is the number of ratios. Therefore

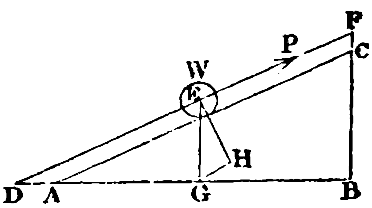
P : force of string $FW :: 1 : 2$ raised to that power. And the force of the string FW is equal to the weight W, because it supports it in the case of equilibrium. Therefore, &c. Q. E. D.

DEF. *The Inclined Plane*, when spoken of as a mechanical power, is a plane supposed to be per-

fectly smooth and hard. It is represented by a line drawn in a vertical plane, and is supposed to pass through this line and to be perpendicular to the vertical plane. A vertical line is supposed to be drawn in the vertical plane from the upper extremity of the inclined plane; and both this vertical line, and the line which represents the inclined plane, are cut by a horizontal line or *base,* drawn in the same vertical plane. The portion of the inclined line and of the vertical line intercepted between the upper point of the plane and its horizontal base, are the *length* and the *height* of the inclined plane respectively.

* Prop. XX. The weight (W) being on an inclined plane, and the force (P) acting parallel to the plane, there is an equilibrium when $P : W ::$ the height of the plane : its length.

Let AC be an inclined plane of which AC is the length, and let W be a weight on the inclined plane, supported by a force P, acting in the direction EF parallel to AC.



The force of the weight W acts in a vertical direction; draw EG vertical to represent this force. Also draw EH perpendicular and GH parallel to the plane AC.

The force EG is equivalent to the two forces EH, HG, (Prop. 13. Cor. 2); of these the force EH is balanced by the reaction of the plane AC, which will balance any force perpendicular to AC, by Axiom 13; and the weight W will be kept at rest if the force HG be counteracted by an equal

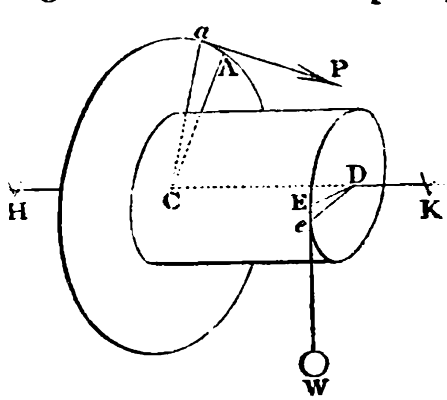
and opposite force P, acting in the direction EF. Therefore there will be equilibrium if P be represented by GH, when W is represented by EG; that is, $P : W :: GH : EG$.

But since EH is perpendicular and GH parallel to the plane AC, EHG is a right angle and therefore equal to ABC. Also the angle EGH is, by parallels, equal to GED, that is, to BFD, that is, to BCA. Therefore the two triangles ABC, EHG, have two angles equal, each to each, and are therefore equiangular, and therefore also similar. Hence $GH : EG :: BC : AC$, and therefore, by what has been proved already, $P : W :: BC : AC$, that is, $P : W ::$ height of plane : length of plane. Q. E. D.

* Def. If two points pass each through a certain space in the same time, the *Velocities* of the two points are to each other in the proportion of these two spaces.

* Prop. XXI. If P and W balance each other on the wheel and axle, and the whole be put in motion, $P : W :: W$'s velocity : P's velocity.

The construction being the same as in Prop. 16, let the machine turn round its axle CD through an angle ACa, or EDe; so that the radius of the wheel at which the power acted, moves out of the position Ca into the position CA; and so that the radius of the axle at which the



power acted, moves out of the position De into the position DE. Then the string by which the power P acts will be unwrapt from the portion aA of the circumference of the wheel, and therefore P will move through a space equal to aA. Also in the same time the string at which W acts will be wrapt upon the axle by a space equal to eE, and therefore W will move through a space equal to eE. Therefore by the definition of velocity aA, eE are as the velocities of P and W.

But since the wheel and axle is a rigid body, turning about the axis CD, all the parts move in planes perpendicular to the axis, and turn through the same angle; and since the plane of the wheel ACa, and of the axle EDe are both perpendicular to the axis, the angles ACa, EDe are the angles through which the radii CA, DE turn. Therefore the angles ACa, EDe, at the centers of the circles ACa, EDe are equal; and therefore, by the Lemma 3, $DE : CA :: Ee : Aa$.

But by Prop. 16, $DE : CA :: P : W$; and by what has been just shewn, $Ee : Aa ::$ W's velocity : P's velocity; therefore $P : W ::$ W's velocity : P's velocity. Q. E. D.

* PROP. XXII. If P and W balance each other in the systems of pulleys described in Propositions 17, 18, and 19, and the whole be put in motion, $P : W ::$ W's velocity : P's velocity.

In Prop. 17, if W be raised through any space, as one inch, the string on each side of the pully A will be liberated for one inch, and therefore P will be at liberty to descend two inches: therefore W's

velocity : P's velocity :: 1 : 2; and since by Prop. 17. $P : W :: 1 : 2$, $P : W ::$ W's velocity : P's velocity.

In Prop. 18, if W be raised through any space, as one inch, each string at the lower block will be liberated one inch, and therefore as many inches of string will be liberated as there are strings at the lower block; and P will be at liberty to descend through a space equal to the whole of this. Therefore the space described by W : space described by P :: 1 : number of strings at the lower block; and hence by Prop. 18, and by the definition of velocity, $P : W ::$ W's velocity : P's velocity.

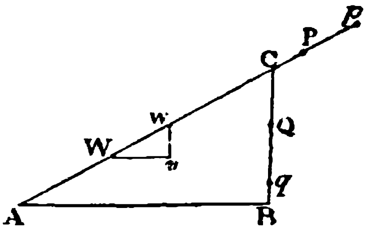
In Prop. 19, if W be raised through any space, as one inch, each of the two strings at the lowest pully E will be liberated one inch; therefore the pully C will be liberated 2 inches, and will rise through 2 inches; therefore on each side the block C, 2 inches of string will be liberated; therefore the pully A will be liberated 2×2 inches; therefore the string on each side the pully A will be liberated 2×2 inches; therefore the string at which P acts will be liberated $2 \times 2 \times 2$ inches, and since this happens at the same time that W is liberated one inch, W's velocity : P's velocity :: $1 : 2 \times 2 \times 2$. And it is clear that the last term is that power of 2 whose index is the number of moveable pulleys.

But by Prop. 19, $P : W :: 1 : 2 \times 2 \times 2$ as before; therefore, by what has been proved, $P : W$ :: W's velocity : P's velocity.

* PROP. XXIII. If P support W on the inclined plane, acting parallel to the plane by means of a string of constant length, and if

the whole be put in motion, $P : W :: W$'s velocity in the direction of gravity : P's velocity.

Let AC be the inclined plane, the weight W being supported by the force P acting parallel to the plane. Let W move to w, and P to p in the same time; and draw Wv horizontal and wv vertical. Then wv is the space through which W moves in the direction of gravity, while P moves through the space Pp, or Ww, which is equal to Pp, because the string wP is always of the same length. Therefore by the definition of velocity, W's velocity in the direction of gravity ; P's velocity :: $wv : Ww$.



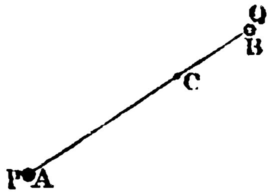
But since Wv is horizontal, or parallel to AB, and wv vertical, or parallel to CB, the triangle Wwv is similar to ACB. Therefore $wv : Ww :: BC : AC$, that is, $wv : Ww ::$ height of the plane : length of the plane. But by Prop. 20, this proportion is that of $P : W$; therefore, by what has been proved $P : W :: W$'s velocity in the direction of gravity : P's velocity.

Cor. If the string by which W is supported pass over a point C and hang vertically, as WCQ, and if Q balance W, Q will descend through a space Qq equal to Ww, when W descends through a space Ww; and we may prove, as before, $Q : W :: W$'s velocity in the direction of gravity : P's velocity.

* Def. The *Center of Gravity* of any body or system of bodies is the point about which the body or the system will balance itself in all positions.

* PROP. XXIV. To find the center of gravity of two heavy points.

Let A, B, be the two heavy points; their weights being P and Q. Join AB; and take in AB a point C, such that $P+Q : Q :: AB : AC$; C will be the center of gravity of A, B.

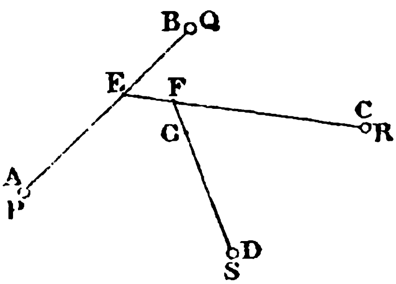


Since $P+Q : Q :: AB : AC$, by division $P : Q :: BC : AC$. Therefore by Prop. 3, A and B will balance each other on the line AB in a horizontal position, because in that case the weights act perpendicularly to the lever. Therefore by Prop. 9, A, B will balance each other on C in every other position of the line AB. Therefore by the definition of the center of gravity, C is the center of gravity of the heavy points A, B.

COR. The pressure upon the center C in every position is equal to $P+Q$, by the Corollary to Prop. 9.

* PROP. XXV. To find the center of gravity of any number of heavy points.

Let A, B, C, D be any number of heavy points; their weights being P, Q, R, S. Join AB, and take a point E in AB, such that $P+Q : Q :: AB : AE$; join EC, and take a point F in EC, such that $P+Q+R : R :: EC : EF$; join FD,



and take a point G in FD, such that $P+Q+R+S : S :: FD : FG$: G will be the center of gravity of P, Q, R, S.

Since $P+Q : Q :: AB : AE$, by Prop. 24, E is the center of gravity of the points A, B, and in every position of AB the pressure upon E is equal to $P+Q$. But since $P+Q+R : R :: EC : EF$, by division $P+Q : R :: CF : EF$; therefore $P+Q$ at E and R at C will balance upon F when EC is horizontal by Prop. 3, and when EC is in any other position by Prop. 9; and the pressure upon F in any position will be $P+Q+R$, by the Cor. to Prop. 9. Therefore in any position P, Q, R will balance upon F, and F is the center of gravity of P, Q, R.

Again, since $P+Q+R+S : S :: FD : FG$, by division, $P+Q+R : S :: DG : FG$; and $P+Q+R$ at F, and S at D, will balance in every position of FD, by Propositions 3 and 9. And the pressure upon G will, in every position of FD, be $P+Q+R+S$, by Cor. to Prop. 9.

Therefore in every position of FD, EC, and BA, the points A, B, C, D will balance upon G; and therefore G is the center of gravity of A, B, C, D.

Cor. 1. It has been shewn that in every position of A, B, C, D the pressure upon G, the center of gravity, is equal to the sum of the weights.

Cor. 2. Every system of heavy points has a center of gravity; for the above construction is always possible.

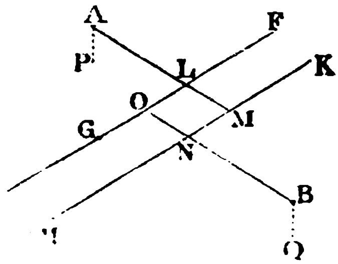
PROP. XXVI. If a straight line pass through the center of gravity of a body, the body will balance itself on this line in all positions.

Since the body will balance itself in all positions upon the center of gravity, if this center be supported the body will be supported in all positions. But if

the line passing through the center of gravity be supported, the center will be supported; and therefore if the line passing through the center of gravity be supported, the body will be supported in all positions; therefore it will balance itself on this line in all positions. Q. E. D.

* PROP. XXVII. **If a system of heavy particles balance upon a straight line in all positions, the center of gravity is in that line.**

Let *HK* be a line on which the system balances itself in all positions; and since every system has a center of gravity (Prop. 25, Cor. 2) if possible let *G*, which is not in *HK*, be the center of gravity.



Let the system be turned round the line *HK* till the plane *GHK* is horizontal; and let *GF* be drawn parallel to *HK*; and let vertical lines be drawn from the particles, meeting the horizontal plane in points as *A*, *B*. Draw *ALM*, *BNO*, perpendicular to *HK* or *GF*, and let *P*, *Q* be the weights of the particles from which the vertical lines fall at *A* and *B* on opposite sides of the lines *GF*, *HK*.

Since the body balances on the line *HK*, the sum of all such moments as $P \times AM$ on the one side of the line *HK* must be equal to the sum of all such moments as $Q \times BN$ on the other side of the line by Prop. 16, Cor. 4. And since, by Prop. 26, the body balances on the line *GF*, the sum of all such moments as $P \times AL$ on the one side of the line *GF* must, for the same reason, be equal to the sum

of all such moments as $Q \times BO$ on the other side of the line GF.

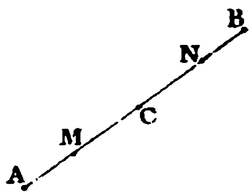
But when we take the moments of the particles of the body with respect to the line GF, instead of HK, each of the moments on the side A, as $P \times AM$, is diminished by $P \times LM$, so as to become $P \times AL$; and each of the moments on the side B, as $Q \times BN$, is increased by $Q \times NO$, so as to become $Q \times BO$: besides which there are particles, the vertical lines from which fall between the lines HK, GF, which are on the side A of the line HK, and on the side B of the line GF; and of which the moments still further diminish the sum of the moments on the side A, and increase the sum of the moments on the side B, when we exchange the line HK ~~GF~~ for the line GF ~~HK~~.

Therefore if the sums of the moments on the sides A and B of the lines HK be equal, the sums cannot be equal when we move the line into the position GF, and therefore by Prop. 16, Cor. 5, the equilibrium cannot subsist for this second line also.

Therefore the point G, out of HK, cannot be the center of gravity; and therefore the center of gravity must be in HK.

* Prop. XXVIII. To find the center of gravity of a material straight line of uniform density.

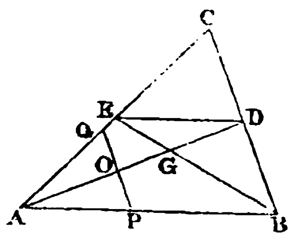
Let AB be the straight line; bisect it in C; C will be the centre of gravity.



Take CM and CN equal, and the line may be considered as composed of pairs of equal particles, placed at points such as M, N, by Axiom 14. But the two particles at M, N balance each other upon the point C in all positions, by Prop. 3 and 9. And all the other pairs of particles

will balance for the like reasons. Therefore the whole line will balance upon C in all positions. Therefore the point C is the center of gravity of the whole line.

* **Prop. XXIX.** To find the center of gravity of a material plane triangle.



Let ABC be the triangle; bisect BC in D and join AD; and bisect AC in E, and join BE; let G be the point of intersection of AD, BE; G is the center of gravity of the triangle.

Draw any line PQ parallel to BC, meeting AD in O; it is easily seen that the triangles AOP, ADB are similar, as also AOQ, ADC.

Hence $OP : OA :: DB : DA$;

and $OA : OQ :: DA : DC$;

therefore $OP : OQ :: DB : DC$.

But DB is equal to DC, therefore OP is equal to OQ, and O bisects PQ.

By Axiom 15, the triangle ABC may be considered as made up of straight lines PQ parallel to BC. And the center of gravity of any one of these lines, as PQ is at O in the line AD: therefore each of these lines will balance upon AD in any position; therefore the whole triangle, which is made up of these lines, will balance upon AD in any position, and therefore the center of gravity of the triangle is in the line AD.

In like manner, the triangle may be considered as made up of straight lines parallel to AC, and it may be proved by similar reasoning that the center of gravity of the triangle is in the line BE.

Therefore the center of gravity of the triangle is at G, the intersection of AD and BC. Q. E. D.

Cor. If we join DE, it is easily shewn that the triangles CBA, CDE are similar; as also AGB, DGE;

therefore $DE : AB :: CD : CB$;

but by construction $CD : CB :: 1 : 2$;

therefore $DE : AB :: 1 : 2$.

Again $GD : AG :: DE : AB$;

therefore $GD : AG :: 1 : 2$;

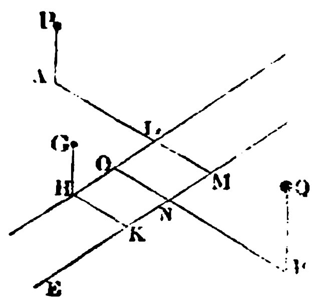
and by composition $AD : AG :: 3 : 2$;

AG is two-thirds of AD, and DG is one-third of AD.

In like manner BG, and GE, are two-thirds and one-third of BC respectively.

Prop. XXX. Any body will have the same effect in producing equilibrium about a horizontal line, as if it were collected at its center of gravity.

Let EF be the horizontal line, and G the center of gravity of the system. Let a horizontal plane be drawn through the line EF, and let GH be a vertical line meeting this plane in H, and PA, GB vertical lines from any particles P, Q of the body, meeting those planes in A, B,



and let HK, AM, BN be drawn perpendicular to EF, and HL parallel to EF.

The effect of the body in producing equilibrium depends upon the excess of the moments such as $P \times AM$, on one side of the line EF, above the moments such as $Q \times BN$, on the other side of the line; and is the same so long as this excess is the same. This follows from Prop. 16, Cor. 3.

Now since G is in the center of gravity, the body

balances on the point G, and therefore on the line HL; for if HL be supported, G is supported. Therefore the sum of all the moments, such as $P \times AL$, on the one side, is equal to the sum of all the moments such as $Q \times BO$, on the other side. And $Q \times BO$ is equal to $Q \times BN + Q \times NO$. Therefore adding $P \times LM$ to both, the sum of moments such as $P \times AL + P \times LM$ or $P \times AM$, is equal to the sum of moments such as $Q \times BN + Q \times NO + P \times LM$. Therefore the excess of moments such as $P \times AM$ over moments such as $Q \times BN$ is the sum of moments such as $Q \times NO + P \times LM$; that is, $Q \times HK + P \times HK$, or $(Q+P) \times HK$; because LM and NO are each equal to HK.

Now if all particles such as P and Q be transferred to G, their effect in producing equilibrium depends upon sums of moments, such as $(P+Q) \times HK$; therefore it is the same as before.

Hence if all the particles P, Q be transferred to the center of gravity G, the effect in producing equilibrium is the same as before. But the whole body may be considered as made up of such particles by Axiom 16. Therefore if a body be collected at its center of gravity, its effect in producing equilibrium will not be altered.

COR. 1. In nearly the same manner it may be proved that any body will exert the same effect in producing equilibrium about any fixed line, as if it were collected at its center of gravity:—namely, by resolving the force arising from the weight of each particle into two component forces; one component force being parallel to the fixed line, and the second component force being perpendicular to the first. The former component forces will not produce any effect to turn the body about the fixed line; and the latter com-

ponent forces will produce the same effect as if the body were collected at the center of gravity, as will appear by comparing the moments in the two cases.

Cor. 2. The effect of the body to disturb equilibrium about a line will be the same as if the body were collected at its center of gravity *G*. For the effect to disturb equilibrium is the effect to produce equilibrium when an adequate force is applied to counteract the tendency to disturb equilibrium.

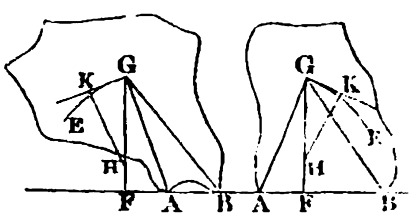
Cor. 3. The effect of a body to produce or disturb equilibrium about a point is the same as if the body were collected at the center of gravity. For any line being drawn through the point, the effect is the same about this line by Cor. 2; and the equilibrium cannot be disturbed about a point, without being disturbed about some line passing through that point.

Def. By the *Base* of a body is meant a side of it touching another body, and on which its direct pressure is supported.

If the body fall over, it tends to turn round one edge of its base, whether the base slide or not.

* Prop. XXXI. When a body is placed upon a horizontal plane, it will stand or fall, according as the vertical line, drawn from its center of gravity, falls within or without its base.

Let *ABCD* be the body, *AB* its base, *G* its center of gravity. First let *GF*, the vertical line drawn from the center of gravity, fall upon the horizontal line *BA* without the base, as at



F. Take in *GF* any line *GH* to represent the weight

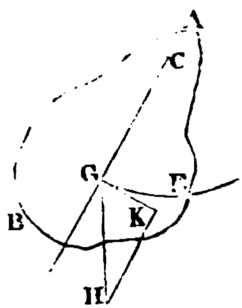
of the body, and draw GK perpendicular to AG and HK parallel to AG.

If the body fall over the edge A of the base, it will tend to turn round the edge A of the base, that is, to describe the arc GE of which the radius is AG. Now by Prop. 30, the effect of the body is the same as if it were collected at the point G. Therefore the force exerted to produce this effect may be represented by the vertical line GH. And the force GH is equivalent to the forces GK, and KH, (acting at G). Of these the force KH acts in the line GA, passing through A, and therefore produces no tendency to motion about A. But the force GK tends to make the body move in the direction GK, which is a tangent to the arc GE; and thus to make the base AB turn round the point A, leaving the plane at B. And there is no force to counteract this tendency; therefore the body will turn round the edge A, on the side on which the perpendicular GF falls.

But if the perpendicular GH fall between A and B, as before, the effect may be represented by the vertical line GH, and the force GH is equivalent to the forces GK, KH. Of these KH (which acts at G) passes through A and does not tend to make the body turn round the edge A; but the force GK, which is a tangent to the arc GE, tends to make the body turn round A in the direction GE. But since the body is rigid, and AB is in contact with the supporting plane, the body cannot turn round the point A in the direction GE, for the pressure thus produced on the horizontal plane is resisted and supported. In like manner the body cannot turn round the edge B by the action of the force GH; therefore in this case the body cannot fall.

* **Prop. XXXII.** **When a body is suspended from a fixed point, it will rest only with its center of gravity in the vertical line passing through the point of suspension.**

Let *AB* be a body suspended from a fixed point *C*, and *G* its center of gravity. If *CG* be not vertical, draw *GH* vertical, and *GK* perpendicular to *CG*, and *HK* parallel to *CG*. The weight of the body will produce the same effect as if it were collected at the point *G*, and may be represented by the line *GH*. But the force *GH* is equivalent to *GK*, *KH*; and of these, the force *KH* (which acts at *G*) is in the line *CG*, and is supported by the fixed point at *C*; and the force *GK* tends to make the body move in *GK*, which is a tangent to *GE*, the path in which the point *G* can move round the fixed point *C*; and there is no force to counteract this tendency, therefore the body will move in this path; and will not rest in the position *AB*.



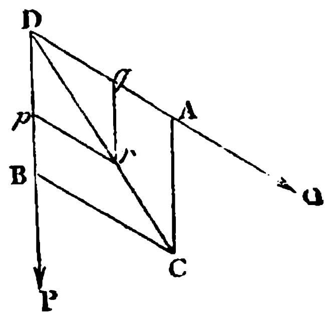
But if *CG* be vertical, the weight will be supported by the fixed point *C*, and there will be no force to produce motion; therefore the body will rest in that position.

Therefore the body will rest only when *CG* is vertical. Q. E. D.

Prop. XXXIII. **If two forces tending to turn a body about a fixed point, and acting in a plane perpendicular to the axis of motion, balance each other, the pressure on the fixed**

point is the same as it would be if the two forces were transferred to the point, retaining their direction and magnitude.

Let P, Q, be two forces, acting to turn a body about a fixed point C. Draw CA parallel to the force P and CB parallel to the force Q; the pressure on C is the same as if the forces P, Q, acted in the lines AC, BC.



Produce the directions of the forces to meet in D, and complete the parallelogram $CADB$. The force P, produces the same effect as if it acted at the point D in P's direction by Axiom 8; and similarly the force Q produces the same effect as if it acted at D. And if Dp, Dq represent the forces P, Q, and the parallelogram $Dprq$ be completed, the diagonal Dr will represent the force at D to which P and Q are equivalent. But the direction of the force Dr must pass through the point C, as in Prop. 12, and will produce the same effect as if it acted at C; and the force Dr acting at C is equivalent to the forces qr, pr, acting in directions parallel to qr, pr, by Prop. 13; that is, the force Dr is equivalent to the forces Dp, Dq, acting in the lines AC, BC; that is, the forces P, Q, acting in the lines BP, AQ are equivalent to forces P, Q acting in AC, BC. Therefore the pressure upon the fixed point C is the same as if the forces P, Q were transferred to that point. Q.E.D.

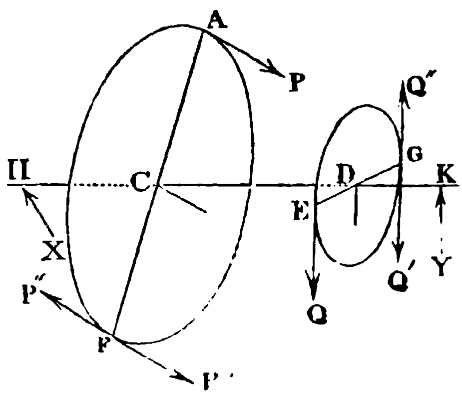
Cor. 1. If, instead of the fixed point at C, we substitute the pressure which that point exerts, there will be equilibrium by Axiom 12. Hence, if a body be acted upon by three forces in the same plane, of which one passes through the intersection of the other two,

and is equal to the resultant of the other two, the body will be in equilibrium.

Cor. 2. Conversely if there be equilibrium, these conditions obtain. This follows from Axiom 3.

Prop. XXXIV. If two forces tending to turn a body round a fixed axis, and acting in two planes perpendicular to the axis, balance each other, (as in the Wheel and Axle,) the pressures upon the points of the axis where the body is supported, are the same as they would be, if the two forces, retaining their direction and magnitude, were transferred to the axis, at the points where the perpendicular planes meet it.

Let P, Q, be two forces acting perpendicularly at the arms CA, DE, to turn a body round the axis HK, the planes CAP, DEQ being perpendicular to HK; and let the forces balance. Let X, Y be the pressures exerted by the fulcrums at H and K, which pressures balance the forces P, Q. Then X and Y are the same as if the forces P and Q, continuing parallel to themselves, were transferred to C and D.



Let AC be produced to F, CF being equal to CA, and at F in the plane PAC, and perpendicular to DG, let two forces P', P'', each equal to P, act in opposite directions. These forces will balance each other and be equivalent to no force; and therefore if the forces

P', P'' are added to the system, the equilibrium will not be disturbed.

In like manner produce ED to G, DG being equal to ED, and at G, in the plane QED, and perpendicular to DG, let two forces Q', Q'', each equal to Q, act in opposite directions: these forces will not disturb the equilibrium. Therefore the six forces P, P', P'', Q, Q', Q'', acting in the manner described, will be supported by the forces X, Y; that is, the eight forces P, P', P'', Q, Q', Q'', X, Y, balance each other.

The forces P'', Q'', are situated in exactly the same manner with regard to vertical lines and planes drawn upwards, as P, Q are, with regard to vertical lines and planes drawn downwards. Therefore P'', Q'', would balance each other on the axis HK, and would produce at H and K pressures equal and opposite to those which P, Q produce. But the forces X, Y are equal and opposite to the pressures which P, Q produce, for they balance those pressures. Therefore the forces P'', Q'' produce at H, K the pressures X, Y.

The forces P, P' are equivalent to a force double of P acting at C, parallel to P; and the forces Q, Q' are equivalent to a force at D double of Q, parallel to Q.

Hence the six forces P, P', P'', Q, Q', Q'' are equivalent to X, Y, at H, K, and to $2P$, $2Q$ at C, D. And the eight forces P, P', P'', Q, Q', Q'', X, Y are equivalent to $2X$, $2Y$ at H, K, and to $2P$, $2Q$, at C, D.

But these eight forces balance each other; therefore $2X$, $2Y$, acting at H, K, balance $2P$, $2Q$, acting at C, D: and therefore X, Y, which balance P, Q, acting at A, E, would balance P, Q, acting at C, D. Q. E. D.

BOOK II. HYDROSTATICS.

DEFINITIONS AND FUNDAMENTAL NOTIONS.

1. HYDROSTATICS is the science which treats of the laws of equilibrium and pressure of fluids.

* 2. Fluids are bodies the parts of which are moveable amongst each other by very small forces, and which when pressed in one part transmit the pressure to another part.

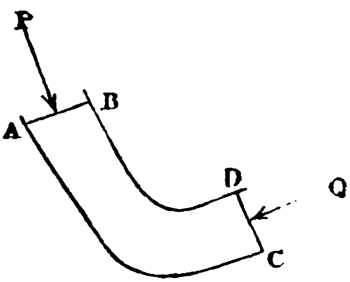
* 3. Some fluids are *compressible* and *elastic*; that is, they are capable of being made to occupy a smaller space by pressure applied to the boundary within which they are contained, and when thus compressed, they resist the compressing forces and exert an effort to expand themselves into a larger space. Air is such a fluid.

* 4. Other fluids are *incompressible* and *inelastic*; not admitting of being pressed into a smaller space nor exerting any force to occupy a larger. Water is considered as such a fluid in most hydrostatical reasonings.

5. In all fluids which have weight, the weight of the whole is composed of the sum of the weights of all the parts.

AXIOMS.

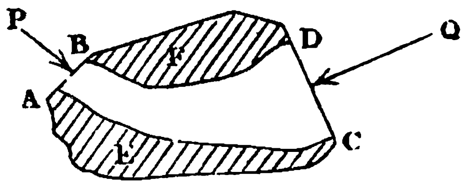
1. If a fluid of which the parts have no weight be contained in a tube of which the two ends are similar and equal planes, two equal pressures applied perpendicularly at the two ends will balance each other.



Let $ABCD$ be the tube, AB, CD its two equal ends: the equal forces P, Q, acting perpendicularly on these ends will balance each other.

2. If a fluid be at rest in any vessel, and if any forces, acting on two portions of the boundary of the fluid, balance each other, they will also balance each other if any portions of the fluid become rigid without altering the magnitude, position, or weight of any of their parts.

Thus if the two forces *P*, *Q*, acting on *AB*, *CD*, parts of the surface of a vessel containing fluid, balance each other; they will also balance each other if



the parts *E* and *F* of the fluid be supposed to become rigid, the magnitude, position and weight, of all the parts of *E*, *F*, remaining unaltered.

3. If two forces acting upon two portions of the boundary of a fluid balance each other, and if a force be added to one of them, it will prevail and drive out the fluid at the part of the surface acted on by the other force.

4. Any surface pressed by a fluid may be divided into any number of particles, and the pressure on the whole is equal to the sum of the pressures on each of the particles.

5. When a plane surface is pressed by a fluid, the pressure exerted on the surface, and the pressure of the surface on the fluid are perpendicular to the plane.

6. We may reason concerning fluids, supposing them to be without weight: and we shall obtain the pressures which exist in heavy fluids, if we add, to the pressures which would take place if the fluids had no weight, the pressures which arise from the weight.

7. When a finite mass of fluid is considered as consisting of small particles of any form or size, and when the consequences of our reasoning do not depend upon the magnitude of the particles, we may, in our reasoning, neglect the magnitude or weight of any single particle, and the consequences will still be true in a heavy fluid.

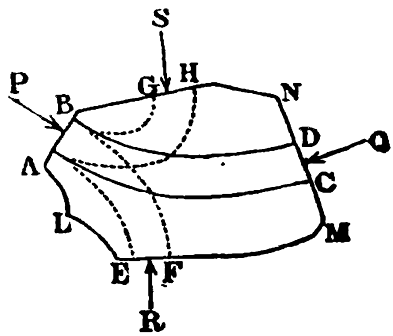
Prop. I. If a fluid without weight be contained in a tube of which the two ends are similar and equal planes, and if two forces applied perpendicularly at the two ends balance each other, the forces are equal.

Let $ABCD$* be the tube, AB, CD its two equal ends; P, Q the two forces. And if P be not equal to Q, let Q be the greater, and let Q be equal to $P+X$. By Axiom 1, the force P acting at AB would balance the force P acting at CD; add to the latter, the force X, and by Axiom 3, the force $P+X$ (that is, the force Q) acting at CD will prevail over the force P acting at AB. Therefore the forces do not balance, which is against the hypothesis. Therefore Q is not the greater of the two P, Q; and in like manner it may be shewn that P is not the greater: therefore P is not unequal to Q, that is, P is equal to Q. Q. E. D.

* **Prop. II.** If a fluid at rest be contained in a close vessel, and if its parts have no weight, on every similar and equal plane portion of the surface of the vessel there will be exerted an equal pressure upon the fluid.

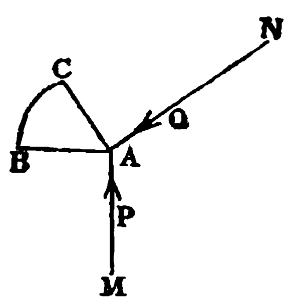
* See figure to Axiom 1.

Let *LMN* be the close vessel, *AB*, *CD*, *EF*, *GH* similar and equal plane portions of the surface of the vessel; let two forces *P*, *Q* acting on *AB*, *CD*, portions of the boundary of the fluid, balance each other; and let a tube *ACBD* be imagined, passing from *AB* to *CD*. Let the portions of the fluid, *ACL*, *BDN* become rigid; then, by Axiom 2, the forces *P*, *Q* still balance each other; but by Prop. 1, in this case the forces *P*, *Q* are equal. And in like manner it may be shewn that the forces *P*, *R* are equal, as also the forces *P*, *S*. And *P*, *Q*, *R*, *S* the forces which act on the boundary of the fluid and balance each other, are the pressures on similar and equal portions of the containing vessel. Therefore the pressures exerted on all such portions are equal. Q. E. D.



* PROP. III. In a fluid at rest, any particle is pressed equally in all directions upon similar and equal plane surfaces.

Let *A* be any point in a fluid, and let *AM*, *AN* be any two directions. Let *AB* be a plane perpendicular to *AM*, and *AC* a similar and equal plane perpendicular to *AN*. Let the solid, of which the planes *AB*, *AC* are boundaries, be completed, and be considered as a particle of the fluid. And let *P*, *Q* be the forces which act on the planes *AB*, *AC*, and preserve the equilibrium. Let the whole of the fluid which sur-



rounds the solid *ABC* be supposed to become rigid: therefore, by Axiom 2, the forces *P*, *Q* still balance each other.

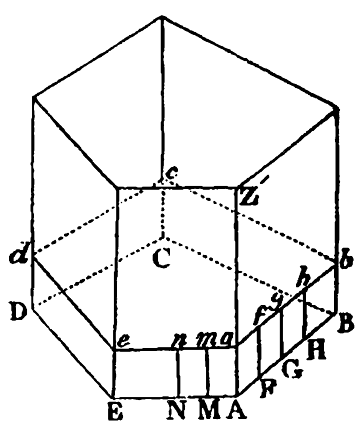
Let the portion *BAC* of fluid have no weight; therefore, by Prop. 2, the forces *P*, *Q* are equal to each other.

But by Axiom 7, since this consequence does not depend upon the magnitude of the particle *ABC*, we may neglect the weight of the particle *ABC*, and the consequence will still be true.

Therefore, in a fluid at rest the pressures *P*, *Q*, which act upon a particle in the two directions *MA*, *NA*, are equal. Q. E. D.

Cor. A particle of fluid is equally pressed on any two equal and similar portions of its surface.

Prop. IV. If a heavy fluid be at rest in a prismatic vessel of which the base is a polygon and the sides vertical, and if a collection of particles of the vertical sides be taken, bounded by two horizontal planes; the pressure on the particles belonging to any part of each side is as the portion of the side.



Let *ABCDE*, *abcde* be the two horizontal planes, and by Lemma 5, the two polygons *ABCDE*, *abcde*, will each be similar and equal in all respects to the polygon which is the base of the vessel. Let *AF*, *FG*, *GH*, be equal portions of the line *AB*, so that the parallelograms *Af*, *Fg*, *Gh* are equal particles of the vertical side *Ab*; and let *AM*, *MN*, be

portions of AE each equal to AF, so that Am, Mn, are equal particles of the side Ae, and Am equal to Af.

Since Af, Fg, may be considered as two equal and similar portions of the surface of a particle, the pressures upon Af and Fg are equal, by Cor. to Prop. 3; for the same reason the pressures upon the particles Fg, Gh, are equal. Therefore the pressures upon all the particles Af, Fg, Gh, are equal, and the whole pressure upon Ah is the same multiple of the pressure upon Af, which AH is of AF.

In like manner, the whole pressure upon An is the same multiple of the pressure upon Am, which AN is of AM.

Therefore,

$AH : AF ::$ pressure upon Ah : pressure upon Af;

and

$AM : AN ::$ pressure upon Am : pressure upon An.

But, by supposition, AM is equal to AF, and by Cor. to Prop. 3, the pressure upon Af is equal to the pressure upon Am.

Therefore, compounding the proportions,

$AH : AN ::$ pressure upon Ah : pressure upon An.

And, in the same manner, we may shew that

$AB : AE ::$ pressure upon Ab : pressure upon Ae;

and that

$AE : DE ::$ pressure upon Ae : pressure upon De.

Therefore, compounding the last two proportions,

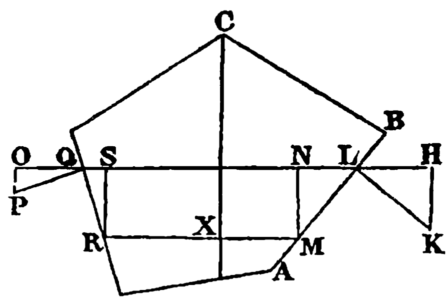
$AB : DE ::$ pressure upon Ab : pressure upon De.

And, in the same manner, the proportions may be proved for any two of the sides, and for any portions of the sides. Q.E.D.

Also, if *AH* be not an exact multiple of *AF*, or *AN* of *AM*, we may, in the reasoning, neglect the last particle in *Ab*, and *Ae*, and the consequence will still be true of a heavy fluid by Axiom 7.

Prop. V. If a heavy fluid be at rest in a vessel of which the base is any horizontal polygon and the sides vertical, the pressures upon the sides of the polygon destroy each other.

Let any collection of particles of the vertical sides be taken, bounded by two horizontal planes, as in Prop. 4. Let *ABC* be one of the horizontal planes, and let any line *CX* be drawn in this plane. In one of the sides *AB* take any portion *LM*, and draw lines *LQ*, *MR*, perpendicular to *CX*, and meeting the opposite side of the vessel in *Q*, *R*. The pressures upon the series of particles of the vertical sides contained between the two horizontal planes, belonging to the lines *LM*, *QR*, are as *LM*, *QR*, by Prop. 4. Take *KL* perpendicular and equal to *LM*, to represent the pressure on *LM*, and then *PQ*, perpendicular and equal to *QR*, will represent the pressure on *QR*. Produce *LQ* both ways, and draw on it perpendiculars *KH*, *PO*. The triangles *KLH*, *LMN* are equal; because *KLM* is a right angle, and therefore the sum of *KLH*, *MLN* is a right angle; but the sum of *KLH*, *LKH* is



a right angle, because *KHL* is a right angle; therefore *MLN* is equal to *LKH*; and the angles at *H* and at *N* are right angles. Therefore the triangles

have two angles of the one equal to two angles of the other; and the side *KL* is equal to *LM*. Therefore the triangles are equal, and *HL* is equal to *MN*; that is, to *RS*. In the same manner it may be shewn that *OQ* is equal to *RS*, that is, to *HL*. Now the force *KL* is equivalent to *KH*, *HL*, and the force *PQ* to *PO*, *OQ*. Therefore the pressures of the two portions of the surface in the directions perpendicular to the line *CX*, are the forces *OQ*, *HL*. But *OQ*, *HL* have been proved to be equal. Therefore those two forces are equal and opposite, and destroy each other. And, in the same manner, if any other two opposite portions of the surface be cut off by lines perpendicular to *CX*, the pressures on these two portions, in a direction perpendicular to *CX*, destroy each other. Therefore the whole of the pressures on the perimeter *ABC*, which are perpendicular to *CX*, destroy each other.

Therefore the forces arising from the pressures of the vertical sides perpendicular to *CX* destroy each other.

In like manner it may be shewn that the forces which arise from the pressure of the vertical sides, and are parallel to *CX*, destroy each other.

Therefore the whole of the forces arising from the pressure of the vertical sides destroy each other. Q. E. D.

SCHOLIUM.

If the base of the figure be curvilinear, and the sides vertical, the same will still be true. For the curvilinear prism is the limit of a polygonal prism of a great number of sides. And what is true up to the limit is true of the limit.

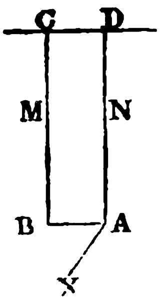
* **Prop. VI.** **If a vessel, the bottom of which is horizontal, and the sides vertical, contain a heavy fluid, the pressure upon the bottom is equal to the weight of the fluid.**

By last Proposition the pressures of the vertical sides destroy each other. Therefore the whole weight of the fluid will be sustained in the same manner as if there were no forces acting on the sides. Let the whole fluid become rigid. Then since it is now a solid body, the pressure upon the base is equal to the weight of the body. But by Axiom 2, the pressure is the same as before; therefore the pressure of the fluid on the base is equal to the weight. Q. E. D.

* Prop. VII. In a fluid of uniform density, the pressure upon a plane surface of any particle, arising from the weight of the fluid, is proportional to the surface pressed, and to its vertical depth below the upper surface of the fluid; provided there is a vertical column of fluid reaching from the particle to the upper surface.

Case 1. When the surface pressed is horizontal.

Let AB be the horizontal surface pressed, and $ABCD$ the column reaching to the surface, the sides AD, BC being vertical.



Suppose the column $ABCD$ to become rigid, the same forces as before still keep it at rest, by Ax. 2. But these forces are the weight of the column acting downwards, and the pressure at the plane AB, acting upwards. Therefore the pressure at the plane AB is equal to the weight of the

column $ABCD$; for, by Prop. 5, the pressure of the surrounding fluid does not increase or diminish the pressure downwards. But the weight of the column AB is, by Lemma 6 and its Corollary, as the base multiplied by the vertical height, that is, as $AB \times AD$. Therefore the pressure which AB exerts is as $AB \times AD$; and therefore the pressure exerted upon AB is also as $AB \times AD$. Q. E. D.

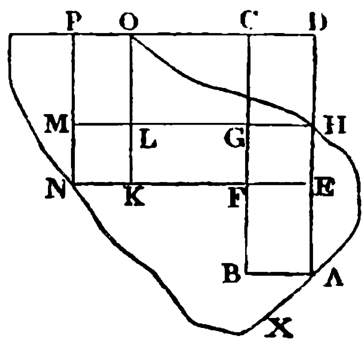
CASE 2. When the surface pressed is not horizontal.

Let AX be another plane surface of the particle, equal to the plane AB. By Prop. 3, the pressure upon AX is equal to the pressure upon AB, and therefore is as $AB \times AC$, or as $AX \times AC$. Q. E. D.

* PROP. VIII. The assertion of the last Proposition is true, when there is not a vertical column reaching from the surface pressed to the upper surface of the fluid.

CASE 1. When the surface pressed is horizontal.

Let AB be the surface pressed, OP the surface of the fluid, OD horizontal and AD vertical.



Draw AH vertical till it meets the side of the vessel; take $HE = AB$, and draw EN horizontal till it meets the opposite side of the vessel; take $NK = AB$, and draw KO vertical; and so on if necessary; we shall in this way arrive at the upper surface of the fluid. Draw HM, NP, so as to com-

plete the zigzag tube $ABEMO$ which passes from the plane AB to the upper surface of the fluid. Also the surfaces EF, FG, KL, LM, are all equal to AB.

In the same manner as in Prop. 7, it appears that the pressure upon ML is as $ML \times LO$, that is, as $AB \times LO$. Therefore supposing the particle MK to be without weight, since KL is equal to LM, by Prop. 2, the pressure upon KL is also equal to $AB \times LO$, and therefore, by Ax. 7, also in a heavy fluid. And since $FGKL$ is a horizontal column of fluid bounded by two equal plane surfaces FG, KL, the pressures exerted by the surrounding fluid on these surfaces are equal, by Prop. 4. Therefore the pressure on FG is as $AB \times LO$. And therefore, since GH is equal to FG, by Prop. 2, the pressure upon GH is as $AB \times LO$; and therefore the pressure of GH downwards is as $AB \times LO$, or $AB \times GC$.

Now by Prop. 5, since AH, BG are vertical, the pressure of the column $ABGH$ downwards is not affected by the surrounding fluid. Suppose $ABGH$ to become rigid, then its pressure downwards will be equal to its weight, together with the pressure on its upper surface GH; that is, it will be as $AB \times AG + AB \times BC$, or $AB \times AC$. Q. E. D.

Case 2. When there is not a vertical column reaching from the surface pressed to the upper surface of the fluid, and the surface pressed is not horizontal.

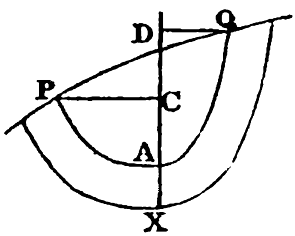
Let AX* be the plane surface pressed, equal to the plane AB. By Prop. 3, the pressure upon AX is equal to the pressure upon AB, and therefore is as $AB \times AD$, or as $AX \times AD$.

* See figure, p. 81.

COR. Hence in all cases the pressure upon a plane surface of any particle is as the surface pressed and as the depth of the particle below the upper surface of the fluid. Q. E. D.

* PROP. IX. The upper surface of a heavy fluid of uniform density, and at rest, is horizontal.

Let PQ be the upper surface of a heavy fluid. If possible, let P, Q not be in a horizontal plane. Let A be any point in the fluid, AX the plane surface of a particle. Draw PC, QD horizontal, and ACD vertical.



By Prop. 8, the pressure upon AX arising from the weight of the fluid is as $AX \times AC$ on the side P; and for the same reason it is as $AX \times AD$ on the side Q: and these are opposite pressures upon the plane AX. Therefore the fluid cannot be at rest except these are equal; that is, except $AX \times AC = AX \times AD$, or $AC = AD$; therefore PQ is not otherwise than horizontal. Q. E. D.

COR. 1. In a heavy fluid at rest, if a horizontal plane $MNOP$* be drawn, and any particle taken, of which the section is AX, the pressures on the two sides of AX, arising from the weight of $MNAX$ and of $POAX$, are equal.

For each pressure is equal to $AX \times AC$, by the Proposition.

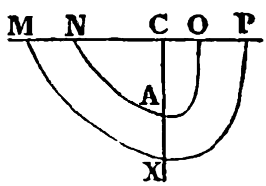
* See figure, p. 84.

Cor. 2. If AX be any plane surface the same is true. For the pressure on any surface is the sum of the pressures on its particles. And the pressures on each particle of AX are equal on the two sides by Cor. 1.

Prop. X. If in a heavy fluid at rest, a horizontal plane be drawn, and equal surfaces taken in this plane, the pressures on them will be equal.

Let $MNOP$ be a horizontal plane in which MN, OP are equal surfaces; the pressures upon MN, OP are equal.

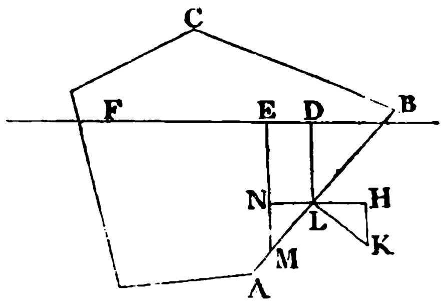
Let a tube $MNOP$ pass from MN to OP, and let AX be a section of it. Suppose the fluid surrounding the tube $MNOP$ to become rigid, the pressures will remain unaltered. But the pressures on AX on one side are (Axiom 6) those arising from the pressure on MN and from the weight of NX; and on the other side the pressures on AX are the pressures arising from the pressure on OP, and the weight of OX; and the pressures on the two sides of AX balance each other and are equal. And of these the pressure arising from the weight of NX is equal to the pressure arising from the weight of OX, by Cor. 2 to last Prop. Therefore, taking away these equals, the remainders, the pressures on MN and OP, are equal. Q. E. D.



*** Prop. XI. If any horizontal prism be partially immersed in a fluid of uniform density, the pressure upwards is equal to the weight of**

the fluid displaced; provided that all the vertical lines drawn from the immersed surface to the upper surface of the fluid are within the prism.

Let ABC be a vertical section of the prism, EF the upper surface of the fluid, LM any particle of one of the surfaces of the prism. Draw LD, ME vertical, meeting the upper surface of the fluid in D and E. Take KL, perpendicular and equal to LM, to represent the pressure on LM, and draw NLH horizontal and KH vertical, meeting NLH.



As in Prop. 5, it may be shewn that the triangles KHL, LNM are equal in all respects, so that KH is equal to LN. The force KL may be resolved into KH, HL, of which KH represents the part which acts vertically upwards; and the whole force on LM is to this part as KL to KH, that is, as LM to LN, or as $LD \times LM$ is to $LD \times LN$. But the whole pressure of the fluid on LM is equal to the weight of the column of fluid $LD \times LM$ (Prop. 7); therefore the part of this pressure which acts vertically upwards is equal to the weight of the column $LD \times LN$; that is, to the weight of the column $LDEN$; that is, to the weight of the column $LDEM$, because we may neglect the weight of the single particle LNM, by Ax. 7.

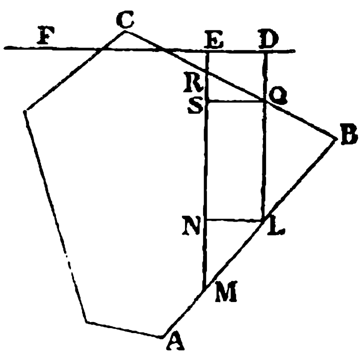
In like manner, the vertical pressure upwards on any other particle of the surface of the prism is equal to the weight of the vertical column of fluid standing

upon that particle and reaching to the upper surface of the fluid. Therefore the whole of the vertical pressures upwards are equal to the sum of all such vertical columns. But the sum of all such vertical columns makes up the fluid displaced; therefore the whole of the vertical pressures are equal to the weight of the fluid displaced. Q. E. D.

Prop. XII. If any horizontal prism be wholly or partially immersed in a fluid of uniform density, the excess of the vertical pressures upwards above the vertical pressures downwards is equal to the weight of the fluid displaced.

Let *ABC* be a vertical section of the prism, *EF* the upper surface of the fluid, *LM* any particle of one of the lower surfaces of the prism.

Draw the column *LDME* vertical, meeting the upper surface of the fluid in *DE*, and cutting off a particle *QR* from the upper surface of the prism. It may be proved, as in the last Proposition, that the vertical pressure upwards on the particle *LM* is equal to the weight of the column of fluid *LDEM*. And in the same manner it may be proved that the vertical pressure downwards on the particle *QR* is equal to the weight of the column of fluid *QDER*. Therefore the excess of the pressures upwards above the pressures downwards on this vertical column is the

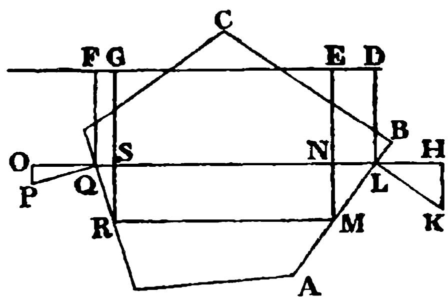


excess of the weight of the column of fluid *LDEM* over that of *QDER*; that is, it is the weight of the column *LQRM*.

In the same manner, in any other vertical column, the excess of the pressure upwards above the pressure downwards is the weight of fluid equal to the vertical column intercepted within the body. And the whole excess of the vertical pressures upwards is the sum of all such intercepted columns; that is, it is the weight of the fluid displaced by the body. Q. E. D.

PROP. XIII. If any horizontal prism be wholly or partially immersed in a fluid of uniform density, the horizontal pressures of the fluid on the sides of the prism destroy each other.

Let *ABC* be a vertical section of the prism, *EF* the upper surface of the fluid; *LM* any particle of one of the surfaces of the prism, and make the same construction as in Prop. 5. Draw *LQ*, *MR* horizontal, cutting off *QR*, a particle of the opposite surface of the prism. Draw *LD*, *ME*, *QF*, *RG*, vertical, to the upper surface of the fluid. Take *KL* perpendicular and equal to *LM* to represent the pressure on *LM*, and draw *NLH* horizontal, and *KH* vertical.



By Prop. 7, the pressures on the particles *LM*, *QR* are as $LM \times LD$ and $QR \times QF$; that is, as *LM* and *QR*, because *LD* and *QF* are equal. Therefore, if a line *KL* equal to *LM* represent the force on *LM*, a line equal to *QR*

will represent the force on QR. Let, therefore, PQ, perpendicular and equal to QR, represent the force on QR, and draw SQO horizontal and PO vertical.

As in Prop. 5, it may be shown that the triangles KHL, LNM are equal in all respects, so that $LH = MN$; also that the triangles POQ, QSR are equal in all respects, so that $OQ = RS$. But MN is $= RS$; therefore $LH = OQ$.

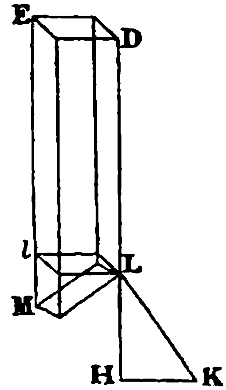
The force KL may be resolved into KH, HL, of which HL is the horizontal part; and the force PQ may be resolved into PO, OQ; of which OQ is the horizontal part; and OQ, HL have been shown to be equal: therefore the horizontal forces on the two particles LM, QR are equal and opposite; therefore they destroy each other.

In the same manner, if any other lines be drawn horizontally in the plane of the figure, they will cut off, in the surface of the prism, opposite particles, on which the horizontal forces will destroy each other; and the horizontal forces on all such particles are the whole horizontal pressures of the fluid on the sides of the prism. Therefore the whole horizontal pressures destroy each other. Q. E. D.

Prop. XIV. If a body bounded by any plane surfaces be wholly or partially immersed in a fluid of uniform density, the pressure of the fluid upwards is equal to the weight of the fluid displaced.

Let LM be a particle of the surface; and on LM let a vertical column be erected, meeting the upper surface of the fluid in DE. Draw the horizontal

section Ll of the column; and take KL perpendicular to LM to represent the pressure on LM, and draw KH perpendicular on the vertical line DL.



The force KL may be resolved into KH, HL, of which HL represents the vertical force; and the whole force on LM is to the vertical force on LM as KL to HL; that is, by Lemma 7, as LM to Ll; or as $DL \times LM$ to $DL \times Ll$. But the whole force on LM is equal to a column of fluid $DL \times LM$, by Prop. 7; therefore the vertical force on LM is equal to a column of fluid $DL \times Ll$; that is, to the column $EDLl$, by Lemma 6; that is, to the column $EDLM$, because the single particle LlM may be neglected.

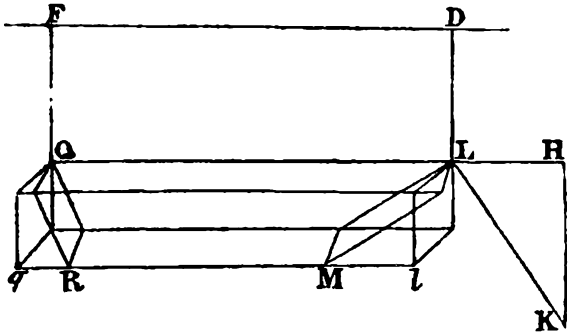
And, in like manner, the vertical pressure upon any other particle of the surface is the weight of fluid equal to the vertical column which stands upon that particle.

And the whole vertical pressure is equal to the sum of all these columns, that is, to the weight of the fluid displaced. Q. E. D.

Also if the fluid be above any part of the body, it may be shown, as in Prop. 12, that the excess of the pressures upwards above the pressures downwards is equal to the weight of the fluid displaced.

PROP. XV. If a body bounded by plane surfaces be wholly or partially immersed in a fluid, the horizontal pressures of the fluid on the sides of the body destroy each other.

Let LM be a particle of the immersed surface of the body, and on LM let a horizontal prism be constituted, (of which QL is one of the edges,) meeting



the opposite surface of the body, and cutting off the particle QR. Draw LD, QF, vertical lines, to the upper surface of the fluid. Take KL to represent the pressure on LM, and KH perpendicular on QL produced. And Let Ll, Qq be sections of the horizontal column by vertical planes.

The force KL may be resolved into KH, HL, of which HL is the horizontal force parallel to the line LQ. And the whole force on LM is to this horizontal force as KL to HL; that is, by Lemma 7, as LM to Ll, or as $LD \times LM$ is to $LD \times Ll$. But the whole pressure on LM is the weight of the column of fluid $LD \times LM$, by Prop. 7. Therefore the horizontal force on LM parallel to LQ is the column $LD \times Ll$.

In like manner, it may be shown that the horizontal force on QR, parallel to QL, is the weight of the column of fluid $QF \times Qq$, which is equal to the column $LD \times Ll$, because, by Lemma 5, Ll, Qq are equal.

Therefore the horizontal pressures on LM and QR, parallel to the line LQ, are equal and opposite, and therefore they destroy each other.

And, in the same manner, the horizontal pressures on any other two opposite particles, parallel to the line *LQ*, destroy each other. And the sum of all such horizontal pressures on opposite particles is the whole pressure on the surface of the body parallel to *LQ*. Therefore the whole of the horizontal pressures parallel to *LQ* destroy each other.

And, in like manner, the whole of the horizontal pressures parallel to any other horizontal line destroy each other.

Therefore the whole of the horizontal pressures destroy each other. Q. E. D.

SCHOLIUM.

The two last Propositions are true of bodies bounded by curvilinear, as well as by plane surfaces. For the curvilinear figure is the limit of a polyhedral figure of a great number of sides. And what is true up to the limit is true of the limit.

* PROP. XVI. When a body floats in a fluid, it displaces as much of the fluid as is equal in weight to the weight of the body; and it presses downwards and is pressed upwards with a force equal to the weight of the fluid displaced.

When a body is immersed in a fluid, by Propositions 12, 13, 14 and 15, and the Scholium, the horizontal pressures destroy each other, and the vertical pressure upwards is equal to the weight of the fluid displaced. But in order that the body may float, the vertical pressure upwards, that is, the weight of the fluid displaced must be equal to the vertical pressure downwards, that is, to the weight of the body. Therefore, when a body floats, it displaces as much, &c. Q. E. D.

Prop. XVII. When a body floats in a fluid the centers of gravity of the body and of the fluid displaced are in the same vertical line.

When a body floats, its weight is balanced by the forces by which it is supported, and by the vertical pressures of the fluid on each particle of the surface; and these latter pressures, by Prop. 14, are equal to the weight of vertical columns which would make up the fluid displaced. And the weights of these vertical columns will produce the same effect as if they were collected at their center of gravity, and acted upwards there; (Book I. Prop. 30), that is, at the center of gravity of the fluid displaced. And the weight of the body produces the same effect as if it were collected at its center of gravity, and acted downwards there. Therefore the two equal forces, one acting vertically upwards at the center of gravity of the fluid displaced, and the other acting vertically downwards at the center of gravity of the body, balance each other. But this cannot be, except they act in the same line; therefore the two centers of gravity are in the same line.

* **Prop. XVIII.** To construct and explain the hydrostatic paradoxes.

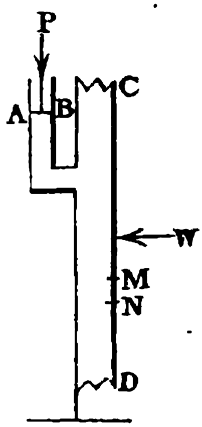
The hydrostatic paradoxes are,

1. That any pressure P, however small, may by means of a fluid be made to balance any other pressure W, however great.

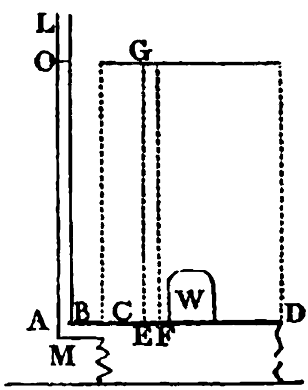
2. That any quantity of fluid, however small, may by means of its weight be made to balance a weight W, however great.

1. The ratio of W to P, however great, may be expressed by a number n.

Let two planes, *AB*, *CD* be taken, such that $1 : n :: AB : CD$; and let a close machine be constructed in which these planes are moveable, so as they can exert pressure on the fluid: as, for example, if *AB* be a *piston*, or plug sliding in a tube, which enters a vessel, and if *CD* be a rigid plane closing a flexible part of the vessel, like the board of a pair of bellows; and let *P* act on *AB*, and let the fluid be in equilibrium. Then the plane *CD* may be divided into *n* surfaces, each (*MN*) equal to *AB*. By Prop. 2, the pressure upon each of these surfaces is *P*, and hence the whole pressure on *CD* is (*Mech.*) the sum of all these pressures: that is, it is *n* times *P*; and if therefore *W* be *n* times *P*, *W* acting at the surface *CD* will be balanced by *P* acting at *AB*.



2. Let the given quantity of fluid be a column of which the base is *B* and the height *H*, and let the weight *W* be equal to *n* times the weight of this column. Take a plane *CD* equal to *n* times *B*, and let a machine be constructed in which there is a vertical tube *LM*, of which the section *AB* is the surface *B*, and which enters a vessel, and *CD* a horizontal plane moveably connected with the vessel, as before. And let the vessel *LMND* be filled with water up to the plane *CD*, and let the weight *W* be placed on the plane *CD*, and the tube *LM* be filled



with fluid to the point O at the height H above CD, so that $ABCD$ being horizontal, AG is equal to H.

The fluid BO and the weight W will balance each other.

For the plane CD may be divided into n particles, as EF, each equal to the plane B; and OG being horizontal, the pressure of the fluid upwards on each of these is equal to a column of fluid $EF \times EG$, or $B \times AO$, or $B \times H$, by Prop. 8. Therefore the whole pressure upwards is n times $B \times H$. Therefore, if the weight W be n times $B \times H$, the pressures downwards and upwards will balance each other, and there will be an equilibrium.

Def. The *Specific Gravity* of a substance is the proportion of the weight of any magnitude of that substance to the same magnitude of a certain standard substance (pure water).

For example, if a cubic foot of stone be three times as heavy as a cubic foot of pure water, the specific gravity of the stone is 3.

The density is as the quantity of matter in a given magnitude, (B. I. Art. 13), and the quantity of matter is conceived to be as the weight: therefore the density of a body is as the specific gravity.

* Prop. XIX. If M be the magnitude of a body, S its specific gravity, and W its weight, W varies as MS.

If the specific gravity increase in any ratio, the weight of a given magnitude increases in the same ratio, by the Definition; that is, the weight W varies as the specific gravity S; also if the specific gravity be given, the weight W increases as the magnitude M; therefore

by the Introduction, Art. 57, if neither S nor M be given, W varies as MS.

Cor. If A be the weight of a unit of magnitude of the standard substance (pure water), $W = AMS$.

For W is equal to MS with some multiplier, whole or fractional, by the Proposition. And when M is 1, and S is 1, by supposition $W = A$; therefore $W = AMS$ in all cases.

SCHOLIUM.

The weight of a cubic foot of water (A) is 63 pounds avoirdupois nearly.

The following is a list of the specific gravity of various substances; the standard (1) being pure water:—

Gold	19.3
Mercury...............	13.6
Silver..................	10.5
Copper	8.9
Lead	11.3
Iron	7.3
Marble	2.7
Water	1.0
Oak	1.2
Fir	.50
Cork	.24
Air	.00125 or $\frac{1}{800}$.

EXAMPLES.

1. To find the weight of a cubic inch of silver.

The formula $W = AMS$ being applied in this case, A is 63 pounds, M is 1 inch, or $\frac{1}{1728}$ foot, S is 10.5;

$$\text{whence } W = \frac{63 \times 10.5}{1728} \text{ pounds} = \frac{661.5}{108} \text{ ounces}$$
$$= 6.1 \text{ ounces.}$$

2. To find the weight of 10 feet square of gold leaf one-thousandth of an inch thick.

$$M = 100 \times \frac{1}{12000}, \; S = 19.3, \; W = \frac{63 \times 19.3}{120} = \frac{1215.9}{120}$$
$$= 10.1 \text{ pounds.}$$

3. To find the weight of a cubical block of marble 1000 feet in the side.

$$W = 63 \times 1000^3 \times 2.7 = 130100000000 \text{ pounds}$$
$$= 58531250 \text{ tons.}$$

4. To find the weight of a column of air one inch base and 5 miles high.

$$W = 63 \times \frac{5 \times 5280}{144} \times .00125 = \frac{63 \times 5 \times 110}{3 \times 800}$$
$$= 14 \text{ pounds.}$$

* Prop. XX. **When a body of uniform density floats on a fluid, the part immersed is to the whole body as the specific gravity of the body is to the specific gravity of the fluid.**

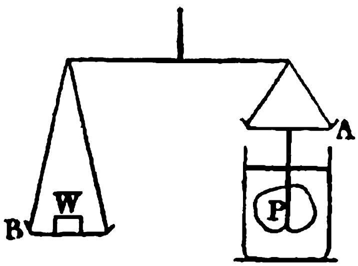
For the magnitude of the part immersed is to that of the whole body as the fluid equal to the part immersed is to the fluid equal to the whole body. But the fluid equal to the part immersed is equal in weight to the whole body, by Prop. 16. Therefore the part immersed is to the whole as the weight of the body is to the weight of an equal bulk of fluid; that is, by the Definition of specific gravity, as the specific gravity of the body to that of the fluid. Q. E. D.

* PROP. XXI. When a body is immersed in a fluid, the weight lost in the fluid is to the whole weight of the body as the specific gravity of the fluid is to the specific gravity of the body.

When the body is wholly immersed, the pressure of the fluid vertically upwards is equal to the weight of a magnitude of fluid equal to the body, Prop. 14. But this pressure upwards diminishes the weight of the body when it is immersed in the fluid, and is the weight lost. Therefore the weight lost in the fluid is equal to the weight of a bulk of fluid equal to the body. And the specific gravity of the fluid is to the specific gravity of the body, as the weight of a bulk of fluid equal to the body is to the weight of the body (Def.); that is, as the weight lost is to the whole weight. Q. E. D.

* PROP. XXII. To describe the hydrostatic balance, and its use in finding the specific gravity of a body heavier than water.

The hydrostatic balance is a balance in which a body (P) can be weighed, either out of water, in the scale A, in the usual manner, or in the water (as at P).



In order to find the specific gravity of any body, let it be weighed out of water, and in water; the difference is the weight lost in water; and hence the specific gravity is known by the last Proposition.

COR. If U be the weight of the body out of water, V the weight in water, W the weight of an equal bulk of water, and S the specific gravity,

$$W = U - V, \text{ and } S = \frac{U}{W} = \frac{U}{U - V}.$$

Def. When a body lighter than water is entirely immersed in water, it tends to ascend by a certain force which is called its *levity*.

* Prop. XXIII. To find the specific gravity of a body lighter than water.

Let the proposed body be weighed out of water; let it be fastened to a *sinker* of which the weight in water is known; and let the compound body be weighed in water.

The excess of the weight in water of the sinker, above the weight in water of the compound body, is the levity of the proposed body: for by attaching the proposed body, its levity or tendency upwards in water diminishes the weight in water of the sinker.

The levity of the proposed body, together with its weight out of water, are equal to the weight of an equal bulk of fluid; for the levity of the body in water is the excess of the pressure upwards above the pressure downwards; that is, the excess of weight of an equal bulk of fluid above the weight (out of water) of the body.

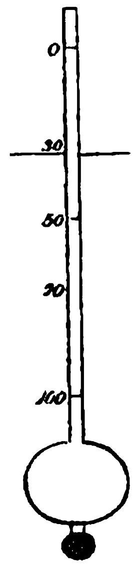
Hence the weight of an equal bulk of water is known, and hence the specific gravity, by the Definition of specific gravity.

Cor. If U be the weight of the body out of water, Q the weight of the sinker in water, and R the weight of the compound body in water, the levity of the compound body is $Q - R$. Hence $Q - R + U$ is the weight of an equal bulk of fluid; and

$$S = \frac{U}{Q - R + U}.$$

* Prop. XXIV. To describe the common hydrometer, and to shew how to compare the specific gravities of two fluids by means of it.

The common *Hydrometer* is an instrument consisting of a body and a slender stem, and of such specific gravity that in the fluids for which it is to be used, it floats with the body wholly immersed and the stem partially immersed.



The part immersed is to the whole as the specific gravity of the body is to the specific gravity of the fluid (Prop. 20); and if the specific gravity of the fluid vary, the part immersed will vary in the inverse ratio of the specific gravity.

But since the stem is slender, small variations of the part immersed will occupy a considerable space in the stem, and will be very easily ascertained.

If the magnitude of the whole instrument be represented by 4000 parts and each of the divisions of the stem by 1 such part; and if the whole length of the stem contain 100 such parts, the instrument will measure with great accuracy specific gravities of fluids within certain limits.

Let the fluids be compared with a certain "proof" standard, as 50, in the middle of the scale. If the instrument sink to 30, the specific gravity of the fluid is known. For the part immersed is $4000-30$, or 3970; and in the "proof" fluid, the part immersed is $4000-50$, or 3950. Therefore the specific gravity of the fluid is to that of "proof fluid" as 3950 to 3970, or as 395 to 397.

PROP. XXV. *INDUCTIVE PRINCIPLE I.* Water and other liquids have weight in all situations.

The facts included in this induction are such as the following:—

(1). Water falls in air as solid bodies do.

(2). A bucket of water held in air is heavy and requires to be supported in the same manner as a solid body.

(3). A bucket of water held in water appears less heavy than in air, and may be immersed so far as not to appear heavy at all.

(4). A lighter liquid remains at rest above a heavier, as oil of turpentine upon water.

(5). The bodies of divers, plants, and other organised bodies, though soft are not compressed or injured under a considerable depth of water.

The different effects (2) and (3) led to the doctrine that all the elements have their *proper places*, the place of earth and heavy solids being lowest, of heavy fluids next above, of light fluids next, of air next; and that the elements do not gravitate when they are in their proper places, as water in water, but that water in air, being out of its proper place, gravitates, or is heavy. In this way also (1) and (4) were explained.

But it was found that this explanation was not capable of being made satisfactory; for—(6) a solid body of the same size and weight as the bucket of water in (3) gave rise to the same results; and these could not be explained by saying that the solid body was in its proper place.

These facts can be distinctly explained and rigorously deduced, by introducing the *Idea* of *Fluid*

Pressure; and the *Principle* that water is a heavy fluid, its weight producing effects according to the laws of fluid pressure.

For on this supposition (1) and (2) are explained, because water is heavy; and (3) is explained by the pressure of the fluid upwards against the bucket, according to Propositions 11, 12, 14.

Also it may be shewn by experiment that in such a case as (4) the lighter fluid increases the pressure which is exerted in the lower fluid.

Facts of the nature of (5) are explained by considering that an equal pressure is exerted on all parts of the organised structure in opposite directions; such pressures balance each other, and no injury results to the structure, except in some cases a general contraction of dimensions. If there be a communication between the fluids within the structure and the fluid in which it is placed, these pressures are exerted from within as well as from without, and the balance is still more complete.

Also all the other observed facts were found to confirm the idea of fluids, considered as heavy bodies exerting fluid pressure: thus it was found—(7) that a fluid presses downwards on a lighter body which is entirely immersed; and presses upwards on a heavier body which is partially immersed; and presses in all directions against surfaces, according to the deductive Propositions which we have demonstrated to obtain in a heavy fluid.

* PROP. XXVI. *INDUCTIVE PRINCIPLE II.* Air has weight.

The facts included in this Induction are such as the following:—

(1). We, existing in air, are not sensible of any weight belonging to it.

(2). Bubbles of air rise in water till they come to the surface.

(3). If we open a cavity, as in a pair of bellows, the air rushes in.

(4). If in such a case air cannot enter and water can, the water is drawn in; as when we draw water into a tube by suction, or into a pump by raising the piston.

(5). If a cavity be opened and nothing be allowed to enter, a strong pressure is exerted to crush the sides of the cavity together.

If facts (1) and (2) were explained at first by saying that the *proper place* of air is above water; that when it is in its proper place, as in (1), it does not gravitate (as in Prop. 25), but that when it is below its proper place, as in (2), it tends to its place; the facts (3) (4) (5) were explained by saying that *nature abhors a vacuum*.

But it was found by experiment:—

(6). That water could not by suction or by a pump be raised more than 34 feet; and stood at that height with a vacuum above it.

(7). That mercury was supported in a tube with a vacuum above it, at the height of 30 inches (Torricelli's experiment).

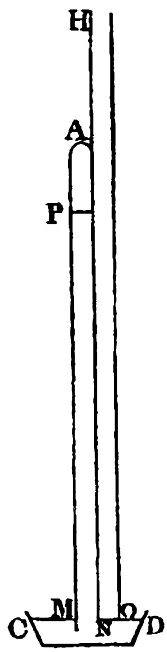
(8). That at the top of a high hill this column of mercury was less than 30 inches (Pascal's experiment).

These facts overturned the explanation derived from nature's horror of a vacuum; for men could not suppose that nature abhorred a vacuum less at the top of a hill than at the bottom, or less over 34 feet of water than over one foot.

But all the facts were distinctly explained and rigorously deduced adopting the *Idea* of fluid pressure, and the *Principle* that air has weight, its weight producing its effects according to the laws of fluid pressure. This will be seen in the Deductive Propositions which we shall demonstrate as the consequences of assuming that air has weight.

The Inductive Proposition was further confirmed by—(9) experiments with the air-pump; for it appeared that as the receiver was exhausted the mercury in the Torricellian experiment fell.

* PROP. XXVII. To explain the construction of the common barometer, and to shew that the mercury in the tube is sustained by the pressure of the air on the surface of the mercury in the basin.



A *Barometer* is a (glass) tube, closed at one end and open at the other, which, being filled with a fluid (as mercury) is inverted with its open end in a basin. In any place the fluid stands at a certain height (if the tube be long enough), leaving a vacuum above.

Since the air has weight, it presses upon the surface *CD* of the mercury in the basin, and this pressure is resisted by the pressure of the column of mercury *PM*, arising from its weight. The mercury in the tube is sustained by the pressure of the mercury in the basin *CD*, which pressure again is sustained by the pressure of the atmosphere on the surface of the mercury in the basin.

* **Prop. XXVIII. In the common barometer, the pressure of the atmosphere is measured by the height of the column of mercury above the surface of the mercury in the basin.**

Let *AM* be the tube, *A* its closed end, *CD* the basin and *MP* the height at which the fluid stands.

The upper parts of the atmosphere are less dense than the lower; but so long as the whole is in equilibrium, this condition does not effect the laws of fluid pressure; and Propositions 1, 2, 3, 4, 5, 6, of this Book will be still true.

Take, on the surface of the basin, *NO* equal to *NM*, the section of the tube; and suppose a tube *HN*, with vertical sides, standing on the base *NO*, to be continued upwards to the limits of the atmosphere. By Axiom 2, if all the rest of the atmosphere became rigid the pressure is not altered; and hence by Prop. 6, the pressure upon *NO* is equal to the weight of the column *HN*. But on this supposition, the pressures on *MN*, *NO* are equal, by Prop. 10. And the pressure on *MN* is equal to the weight of the vertical column of mercury *MP*. Therefore the weight of the column of mercury is equal to the weight of the column of atmosphere on the same base. Therefore the weight of the column of atmosphere is measured by the weight of the column of mercury; that is, the pressure of the atmosphere on a surface equal to the section of the tube made at the surface of the mercury in the basin, is equal to the weight of the vertical column of mercury which stands on the same section.

Therefore the pressure of the atmosphere is measured by the weight of the column of mercury, that is, by the height, if the section and the density con-

tinue constant; for the weight of a column is as section × height × density.

Cor. 1. If, instead of mercury, the tube be filled with any other fluid, as water, the fluid will stand at such a height as to support the weight of the atmosphere; and the height will be greater as the density of the fluid is less.

The mean height of the mercury-barometer being 30 inches, and the specific gravity of mercury 13.6, the mean height of the water-barometer is 13.6×30 inches $= 408$ inches $= 34$ feet.

Cor. 2. If the tube AM be not vertical, the proposition is still true, the vertical height of A above M being still taken for the height of the fluid; for the pressure on MN is the same as if AM were vertical, by Prop. 8.

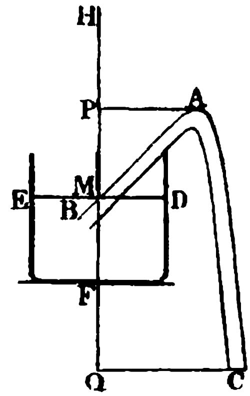
Cor. 3. If the portion of the tube AP, instead of being a vacuum, contain air of less density than the atmosphere, a column of fluid PM will still be sustained, smaller than the column where AP is a vacuum; for if P were to descend to M, the pressure on MN would be less than the pressure on NO, which is impossible.

* Prop. XXIX. To describe the siphon and its action.

A *Siphon* is a bent tube, open at both ends, and capable of being placed with one end in a vessel of fluid, and the other end lower than the upper surface of the fluid in the vessel.

Let BAC be the bent tube placed so that the end B is immersed in the water FED, and the outer end C is below the surface ED.

If the tube BAC be filled with water, and if the vertical height of the portion MA be less than the height of the water-barometer, the tube will act as a siphon, that is, the water will constantly run through the tube BAC and out at C.



The tube being filled with water, let the end C be stopped; and let HM be the height of the water-barometer; AP, CQ horizontal. The pressure which acts upwards on the column MA at M is equal to the column of the water-barometer HM (Prop. 10), and the pressure downwards which arises from the weight of the fluid AM is equal to a vertical column PM (Prop. 8); therefore the remaining pressure urging the fluid in the tube in the direction BAC is equal to a column of water HP; also the weight of the fluid in AC urges the water in the same direction, with a force equal to a column PQ (Prop. 8); therefore the obstacle at C sustains a pressure downwards equal to a column $HP + PQ$ or HQ. But the pressure on C upwards is equal to the column of the water-barometer HM: therefore the remaining pressure downwards at C is equal to the remaining column MQ.

And if there be no obstacle at C, the fluid in the siphon BAC will be urged in the direction BAC by a force equal to a column of fluid MQ.

But if the vertical height of MA be greater than that of the water-barometer, there will be a vacuum formed above the fluid at A (Prop. 28, Cor. 1), and the siphon will not act.

Also, if instead of water and the water-barometer, we had taken any other fluid and the corresponding

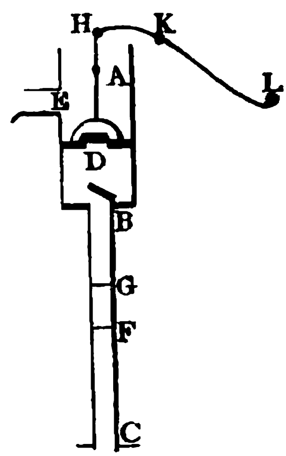
barometer, the reasoning, and the result, would have been the same as above.

* Prop. XXX. To describe the construction of the common pump, and its operation.

A *valve* is an appendage to an orifice closing it and opening in such a manner as to allow fluid to pass through the orifice in one direction and not in the opposite one.

A *piston* is a plug capable of sliding in an orifice or tube so as to produce or remove fluid pressure.

The *Common Pump* consists of a cylindrical *barrel* AB, closed at bottom with an upwards-opening valve B, and of a *piston* D with an upwards-opening valve, which moves up and down in the barrel. A *suction-pipe* BC passes downwards from the valve B to the *well* at C, and the water which rises above the piston is delivered by the *spout* E.



The operation of the pump is as follows. The piston D being in its lowest position, is raised to its highest position by means of the lever HKL. Since the valve D opens upwards, no air is admitted at D during this rise; and since the valve B opens upwards, the air which occupied CD follows the piston in its ascent; it expands, and its pressure on the water at C is diminished. Hence the water in the suction-pipe rises by the pressure of the atmosphere on the surface of the well to some point F. (Cor. 3 to Prop. 28).

The piston is then made to descend to its lowest position, the valve B is closed, and therefore the quan-

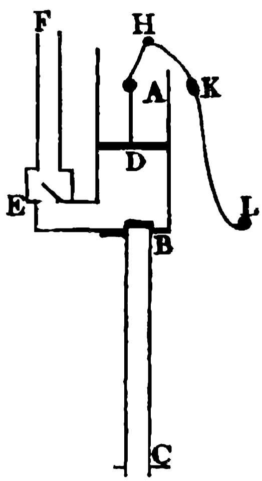
tity of air in FB is not changed, and the water remains at F, while the air in BD escapes by the valve at D.

The piston is then again raised, the air in DF expands as before, and the surface of the water at F comes to a new position at G.

The same movements being repeated, the water will again rise; and so on, till it reaches the piston D, after which time the piston in its ascent will lift the water, and when it has lifted it high enough, will deliver it out at the spout E.

* PROP. XXXI. To describe the construction of the forcing pump and its operation.

The *Forcing Pump* consists of a cylindrical *barrel* AB, closed at bottom with an upwards-opening valve B; of a *piston* D with no valve; and of a *spout* E with an outwards-opening valve. The piston moves up and down, and the suction-pipe descends from the bottom of the barrel to the well, as before, and the spout carries the water upwards.



The operation of the pump is as follows. The piston D, in ascending from its lowest to its highest position, draws the water after it as in the common pump. When the piston descends, the air is forced out at the valve E; and after a certain number of ascents, the water comes into the barrel AB. When the piston next descends it forces the water through the valve E, and continues afterwards to draw the water through the valve B in its rise, and to extrude it through the valve E in its descent.

* Prop. XXXII. *Inductive Principle III.*

Air is elastic; and the elastic force of air at a given temperature varies as the density.

The facts which shew air to be elastic are such as follow:—

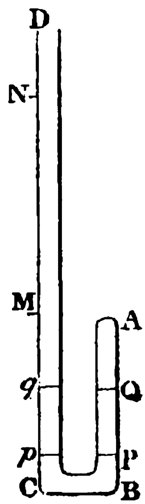
(1). A bladder containing air may be contracted by pressure, and expands again when the pressure is removed.

(2). A tube closed above and open below, and containing air, being immersed in water, the air contracts as the immersion is deeper, and expands again when the tube is brought to the surface.

(3). If a close vessel containing water and air, fitted with a tube making a communication between the water and the exterior, be placed in the exhausted receiver, the water is expelled through the tube.

The principle that the elastic force increases *in proportion to* the density, was experimentally proved (first by Boyle*) in the following manner:—

A uniform tube $ABCD$ was taken, closed at A and open at D, and bent so that BA and CD were upright at the same time. Quicksilver was poured in, so that its ends stood at M and P. Again, more quicksilver was poured in, so that its ends stood at N and Q. And Pp, Qq being horizontal, it appeared that when AP was double of AQ, Nq was double of Mp, and so on for any other proportion; so that generally



$$AP : AQ :: Nq : Mp.$$

Let A be the horizontal section of the tube at AP; and by Prop. 8, the pressure of the fluid on this section is $A \times Mp$,

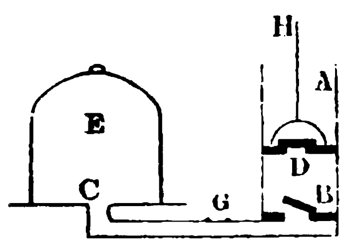
* Shaw's Boyle, Vol. II. p. 671.

and this pressure is balanced by the elastic force of the air in AP, and is therefore equal to it; and the pressure upon the section at Q is $A \times Nq$, which is, in like manner, equal to the elastic force of the air in AQ. Hence the elastic force of the air in AQ is to the elastic force of the air in AP as $A \times Nq$ to $A \times Mp$; that is, as $A \times AP$ to $A \times AQ$; that is, inversely as the space occupied.

The quantity of air remaining the same, the density is inversely as the space occupied: therefore the elastic force is as the density.

* PROP. XXXIII. To describe the construction of the air-pump and its operation.

The *Air-pump* consists of a barrel and piston with valves, like a common water-pump; the suction-pipe communicating with a close vessel called the *receiver*.



Let AB be the barrel, B the inwards-opening valve at the bottom of the barrel, D the piston with its outwards-opening valve, BC the pipe, E the receiver.

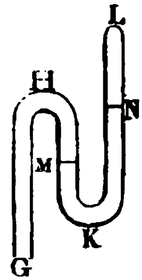
The piston D being in its lowest position, is raised to its highest position by the handle H. During the rise no air is admitted at D, and the air in CD, by its elasticity, follows the piston in its ascent, passing through the valve B, and thus air is drawn out of the receiver E.

The piston is then made to descend again to its lowest position: no air returns through the valve B, and the air in BD escapes by the valve at D.

The piston is again raised, and more air is drawn out of E as before: and so on without limit.

PROP. XXXIV. To explain the construction of the siphon-gauge.

The *Siphon-gauge* is a bent tube, closed at one end, and containing fluid, fixed to an air-pump C or other machine, to determine the degree of rarefaction of the air.

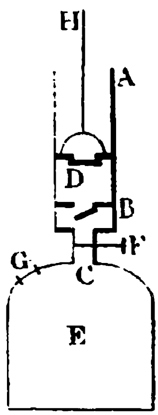


Let *GHKL*, closed at *L*, be the siphon-gauge, (fixed to *G* in the last Prop.), and let *MKN* be a portion of the tube filled with mercury, *LN* being a vacuum. Then the vertical height of *N* above *M* measures the density of the air in *GHM*.

If *GHM* were a vacuum (that is, if the exhaustion in the air-pump were complete) *M*, *N* would be at the same level.

* PROP. XXXV. To describe the condenser and its operation.

The *Condenser* consists of a barrel and piston with valves, opening the contrary way from those of the common water-pump, and communicating by a pipe with a closed receiver.



Let *AB* be the barrel, *B* the inwards-opening valve at the bottom of the barrel, *D* the piston with its inwards-opening valve, *BC* the pipe, *E* the receiver.

The piston *D* being in its highest position, is forced to its lowest position by the handle *H*. During the descent no air escapes through *D*, and the air in *BD* is driven through the valve *B*, and increases the quantity in the receiver *E*.

The piston is then made to ascend, and no air escapes at *B*, because the valve opens inwards; but air enters the barrel *BD* by the valve *D*.

The piston is again forced down, and more air is driven into the receiver *E* as before: and so on without limit.

The pipe *BC* has a stop-cock *F*, and when this is closed, the pump may be screwed off, after the condensation is made.

*Prop. XXXVI. *Inductive Principle IV.*

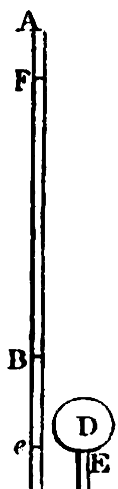
The elastic force of air is increased by an increase of temperature.

The facts included in this induction are such as the following:—

(1). If a bladder partly full of air be warmed it becomes more completely full.

(2). If an inverted vessel confining air in water be warmed the air escapes in bubbles.

It was experimentally ascertained *how much* the elastic force of air is increased by heat (first by Amontons*), in the following manner:—



A bent tube *ABC*, with a bulb *D* containing common air, was filled with mercury from *B* to *E*, *B* being a little higher than the horizontal plane *Ee*. The bulb was then placed in boiling water, and it was found that a small portion of the mercury was driven out of the bulb, so that the extremity of the column was elevated to *F*, *BF* being nearly 10 inches.

* Mem. de l'Acad. Roy. des Sciences de Paris. 1699. p. 113.

The air occupies very nearly the same space in the last case as in the first; for the bore of the tube was very small, and the surface of the mercury continued nearly in the same position at E. Hence BF is the increase of the column eF, which measures the pressure of the air in D. But the pressure on D is the elasticity of common air, or the pressure of the atmosphere, which is about 30 inches. Therefore of mercury in this experiment the elasticity was increased from 30 to 30 + 10, by heating the water to boiling: that is, the elasticity was increased about one third.

PROP. XXXVII. *INDUCTIVE PRINCIPLE V.* Many (or all) fluids expand by heat; and the amount of expansion at the heat at which water boils, and at the heat at which ice melts, are each a fixed quantity.

The former part of this proposition is proved by including the fluids in bulbs, which open into a slender tube; for a small expansion of the fluid in the bulb is easily seen, when it takes place in the slender tube.

It was at first supposed, that when a fluid is exposed to heat, (as, for instance, when a vessel of water is placed on the fire,) a constant addition of heat takes place, increasing with the time during which the fire operates.

But it appeared, that when a tube containing air is placed in water thus exposed to heat, the expansion of the air (observed in the way described in Prop. 36) goes on till the fluid boils, after which no additional expansion takes place.

This fact is explained by assuming the expansion of air as the *Measure* of heat, and by adopting the

Principle that the heat of boiling water is a fixed quantity.

This principle was first experimentally established by Amontons. Afterwards it was ascertained by Fahrenheit (1714), and others, that the expansion of oil, spirit of wine, mercury, at the heat at which water boils, is a fixed quantity; and hence Fahrenheit made the *boiling point* of water one of the fixed points of his thermometers, which were filled with spirit of wine or with mercury.

For another fixed point he took the cold produced by a mixture of ice, water, and salt; and he assumed this to be the *point of absolute cold.*

But it was found by Reaumur (1730), that the *freezing point* of water, or the melting point of ice, is more fixed than the point of absolute cold determined in the above manner. This was proved in the same manner in which the heat of boiling water had been proved to be a fixed point. The *freezing point* was then adopted as one of the fixed points of the measure of heat.

* Prop. XXXVIII. To show how to graduate a common thermometer.

The common *Thermometer* is an instrument consisting of a bulb and a slender tube of uniform thickness, containing a fluid (as mercury or spirits of wine) which expands by heat and contracts by cold, so that its surface is always in the tube.

Let the instrument be placed in boiling water, and let the point to which the surface of the fluid expands in the tube be marked as the *boiling point.*

Let the instrument be immersed in melting ice, and let the point to which the surface of the fluid contracts in the tube be marked as the *freezing point.*

For *Fahrenheit's division*, divide the interval between the freezing point and the boiling point into 180 equal parts; and continue the scale of equal parts upwards and downwards. Place 0 at 32 parts below the freezing point, 32 at the freezing point, 212 at the boiling point; and the other numbers of the series at other convenient points, and the scale is graduated, the numbers expressing degrees of heat according to the place of the surface of the fluid in the tube.

For the *centigrade division*, divide the interval between the freezing and boiling point into 100 equal parts; mark the freezing point as 0 degrees, the boiling point as 100 degrees, and so on as before.

* PROP. XXXIX. To reduce the indications of Fahrenheit's thermometer to the centigrade scale, and the converse.

To reduce Fahrenheit to centrigrade, subtract 32, which gives the number of degrees above the freezing point: and multiply by $\frac{5}{9}$, because 180 degrees of Fahrenheit are equal to 100 centigrade.

Thus

$$59^0 F = 27^0 F \text{ above } 32^0 F = \frac{5 \times 27^0}{9} \text{ centig. above } 0.$$
$$= 15^0 \text{ cent.}$$

To reduce centigrade to Fahrenheit, multiply by $\frac{9}{5}$, which gives the number of Fahrenheit's degrees above the freezing point, and add 32, which gives the number above Fahrenheit's zero.

Thus

$$60 \text{ centig.} = 90 F \text{ above freezing} = 90 F + 32 F = 122 F.$$

BOOK III. THE LAWS OF MOTION.

DEFINITIONS AND FUNDAMENTAL PRINCIPLES.

1. The science which treats of Force producing Motion, and of the Laws of the Motion produced, is Dynamics.

2. In Dynamics, we adopt the Ideas, Definitions, Axioms, and Propositions of Statics.

3. We require also several new Ideas, Definitions, and Principles, which are obtained by Induction, and will be stated in the succeeding Propositions.

4. Velocity is the degree in which a body moves quickly or slowly: thus, if a body describes a greater space than another in the same time, it has a greater velocity.

5. The velocity of a body is *uniform* when it describes equal spaces in *all* equal times.

6. The velocities of bodies, when uniform, are *as* the spaces which they describe in equal times.

Def. 1. The velocity of a body moving uniformly is *measured* by the space described in a unit of time.

When the velocities of bodies are not uniform, they are increasing or decreasing.

Axiom 1. If a body move with an *increasing* velocity, the space described in any time is *greater* than the space which would have been described in the same time, if the velocity had continued uniform for the same time, and the same as it was at the *beginning* of that time.

And the space described in any time is *less* than the space which would have been described in the

same time, if the velocity had been uniform for the same time, and the same as it is at the *end* of that time.

Axiom 2. If a body move with a *decreasing* velocity the above Axiom is true, putting "less" for "greater," and "greater" for "less."

Axiom 3. If two bodies move, having their velocities at every instant in a constant ratio, the space described in any time by one body and by the other will be in the same ratio.

Axiom 4. If several detached material points, acted upon by any forces, move in parallel lines parallel to the forces in such a manner as to retain always the same distances from each other and the same relative positions, they may be supposed to be rigidly connected, and acted upon by the same forces, and their motions will not be altered.

Axiom 5. On the same suppositions, the parallel forces may be supposed to be added together so as to become one force, and the motions will not be altered.

Axiom 6. When bodies in motion exert pressure upon each other, by means of strings, rods, or in any other way, the reaction is equal and opposite to the action at each point.

Definition 2 (of Force), Def. 3 (of the Direction of Force), stand after Prop. 3; Def. 4 (of Uniform Force), stands after Prop. 3; Def. 5 (of Composition of Motions), after Prop. 8; Def. 6 (of Accelerating Force), after Prop. 13; Def. 7 (of Momentum), Def. 8 (of Elastic and Inelastic Bodies), Def. 9 (of Direct Impact) after Prop. 17.

Axiom 7 stands after Prop. 2; Axiom 8 and 9 after Prop. 17.

Prop. I. In uniform motion, the space described with a velocity v in a time t is tv.

For (Def. 1.) v is the space described in each unit of time, and t the number of units; therefore the whole space described is tv.

Prop. II. *Inductive Principle I. First Law of Motion.*

A body in motion, not acted upon by any force, will go on for ever with a uniform velocity.

The facts which are included in this induction are such as the following:—

(1). All motions which we produce, as the motions of a body thrown along the ground, of a wheel revolving freely, go on for a certain time and then stop.

(2). Bodies falling downwards go on moving quicker and quicker as they fall farther.

It was attempted to explain these facts, by saying that motions such as (1) are *forced* motions, and motions such as (2) are *natural* motions; and that forced motions decay and cease by their nature, while natural motions, by their nature, increase and become stronger.

But this explanation was found to be untenable; for it was seen—(3) that forced motions decayed less and less by diminishing the obvious obstacles. Thus a body thrown along the ground goes farther as we diminish the roughness of the surface; it goes farther and farther as the ground is smoother, and farther still on a sheet of ice. The wheel revolves longer as we diminish the roughness of the axis; and longer still, if we diminish the resistance of the air by putting the wheel in an exhausted receiver.

Thus a decay of the motion in these cases (1) is constantly produced by the obstacles. Also an increase of the motion in the cases (2) is constantly produced by the weight of the body.

Therefore there is in these facts nothing to show that any motion decays or increases by its nature, independent of the action of external causes.

(3). By more exact experiments, and by further diminishing the obstacles, the decay of motion was found to be less and less; and there was in no case any remaining decay of motion which was not capable of being ascribed to the remaining obstacles.

Hence the facts are explained by introducing the *Idea* of *force*, as that which causes change in the motion of a body; and the *Principle*, that when a body is not acted upon by any force, it will move with a uniform velocity.

Cor. 1. When a body moves freely (not being retained by any axis or any other restraint), and is not acted upon by any force, it will move in a straight line.

For since it is not acted on by any force, there is nothing to cause it to deviate from the straight line on any one side.

Def. 2. *Force* is that which causes change in the state of rest or motion of a body.

Def. 3. When a Force acts upon a body, and puts it in motion, the line of direction of the motion is the *direction of the force*.

Axiom 7. When a Force acts upon a body in motion, so that the direction of the force is the direction of the motion, the force will not alter the direction of the motion.

PROP. III. *INDUCTIVE PRINCIPLE II. Gravity is a uniform force.*

The facts which are included in this induction are such as the following:—

(1). Bodies falling directly downwards fall quicker and quicker as they descend.

It was inferred, as we have seen in the last proposition, that the additions of velocity in the falling bodies are caused by gravity.

An attempt was made to assign the law of the increase of velocity conjecturally, by introducing the Definition, that a uniform force is a force which, acting in the direction of a body's motion, adds equal velocities in equal *spaces*, and the Proposition that gravity is a uniform force.

The Definition is self-contradictory. But if it had not been so, the Proposition could only have been confirmed by experiment.

(2). It appeared by experiment that when bodies fall (down inclined planes) the spaces described are as the squares of the times from the beginning of the motion.

This was distinctly explained and rigorously deduced by introducing the *Definition* of *uniform force;* that it is a force which, acting in the direction of the body's motion, adds equal velocities in equal *times;*

And the *Principle* that gravity (on inclined planes) is a uniform force.

For it may be proved deductively, as we shall see, that this definition being taken, the spaces described in consequence of the action of a uniform force are as the squares of the times from the beginning of the motion. And if the force be other than uniform, the spaces will not follow this law. Therefore the Proposition, that

gravity on inclined planes is a uniform force, is the only one which will account for the results of experiment.

Also if the force of gravity on inclined planes be a uniform force, the force of gravity when bodies fall freely is uniform, for when the inclined plane becomes vertical, the law must remain the same.

(3). The Proposition is further confirmed by shewing that its results, obtained deductively, agree with experiments made upon two bodies which draw each other over a fixed pully (Atwood's Machine); and—(4) by the times of oscillation of pendulums.

Also it appears that when gravity acts in a direction opposite to that of a body's motion, it subtracts equal velocities in equal times.

Hence we introduce the following Definition.

DEF. 4. A *uniform force* is that which, acting in the direction of the body's motion, adds or subtracts equal velocities in equal times.

PROP. IV. If a uniform force act upon a body moving it from rest, and if a be the velocity at the end of a unit of time, v, the velocity at the end of t units of time, is ta.

For the body will move in the direction of the force (Def. 3), and therefore the force is in the direction of the motion; and therefore by Axiom 7, the direction of the motion is not altered by the action of the force. Hence by Def. 4, the velocity added to the velocity in each second is a, and in t seconds from the beginning of the motion it is ta.

COR. 1. At the end of $\frac{1}{n}$ of a unit of time the velocity is $\frac{a}{n}$.

Cor. 2. At the end of $\frac{m}{n}$ units of time, the velocity is $\frac{ma}{n}$.

Cor. 3. If v be the velocity at the end of the time t, the velocity at the end of the time $\frac{m}{n}t$ will be $\frac{m}{n}v$.

Prop. V. If a uniform force act upon a body moving it from rest, and if a be the velocity at the end of a unit of time, s, the space described at the end of t units of time, is $\frac{1}{2}at^2$.

Let each unit of time be divided into n equal portions; each of these will be $\frac{1}{n}$; and the whole number will be tn; and the velocity at the *beginning* of the first, second, third, fourth, &c. of these portions will be, by Prop. 1, Cor. 2,

$$0,\ \frac{a}{n},\ \frac{2a}{n},\ \frac{3a}{n},\ \text{\&c. } (tn \text{ terms}).$$

Suppose spaces to be described in these portions of time with the velocity at the beginning of each portion continued uniform during that portion; these spaces are by Prop. 1,

$$0 \times \frac{1}{n},\ \frac{a}{n} \times \frac{1}{n},\ \frac{2a}{n} \times \frac{1}{n},\ \frac{3a}{n} \times \frac{1}{n}\ (tn \text{ terms})$$

which form an arithmetical series. And the last term of this series is

$$\frac{(tn-1)a}{n} \times \frac{1}{n}.$$

And the sum of it is (Introd. Art. 60)

$$\frac{(tn-1)a}{n} \times \frac{1}{n} \times \frac{tn}{2};$$

$$\text{or } \frac{(tn-1)at}{2n} \text{ or } \frac{at^2}{2} - \frac{at}{2n}.$$

In the same manner the velocity at the *end* of the first, second, third, &c. of these portions is

$$\frac{a}{n}, \frac{2a}{n}, \frac{3a}{n}, \text{ \&c. } (tn \text{ terms}).$$

Suppose spaces to be described in these portions of time with the velocity at the end of each portion continued uniform during the time. These are as before

$$\frac{a}{n} \times \frac{1}{n}, \frac{2a}{n} \times \frac{1}{n}, \frac{3a}{n} \times \frac{1}{n}, \frac{4a}{n} \times \frac{1}{n} \ (tn \text{ terms});$$

an arithmetical progression, of which the last term is $\frac{tna}{n} \times \frac{1}{n}$, and the sum is $\left(\frac{tna}{n} \times \frac{1}{n} + \frac{a}{n} \times \frac{1}{n}\right)\frac{tn}{2}$,

$$\text{or } \frac{(tn+1)at}{2n} \text{ or } \frac{at^2}{2} + \frac{at}{2n}.$$

But in this case the body moves with a constantly increasing velocity. Therefore by Axiom 1, the space described in each of the times $\frac{1}{n}$ is greater than the space described on the former of the above suppositions; and less than the space described on the latter of the above suppositions. Hence the whole space s is greater than $\frac{at^2}{2} - \frac{at}{2n}$, and less than $\frac{at^2}{2} + \frac{at}{2n}$.

Therefore it is equal to $\frac{at^2}{2}$: for if not, let it be equal to a greater quantity, as $\frac{at^2}{2}+b$, and let $n=\frac{at}{2b}$: then $\frac{at}{2n}=b$; and therefore the space described is equal to $\frac{at^2}{2}+\frac{at}{2n}$: but it is less; which is impossible. Therefore the space is not equal to a greater quantity than $\frac{at^2}{2}$; and in like manner it may be shewn that it is not equal to a less. Therefore the space is equal to $\frac{at^2}{2}$. Q. E. D.

Cor. Hence if t, T, be any two times from the beginning of the motion and s, S the spaces through which a body falls in those times, $s : S :: t^2 : T^2$.

Prop. VI. The space described in any time from rest by the action of a uniform force is equal to half the space described by the last acquired velocity continued uniform for the time.

As in last Proposition, let t be the whole time, and a the velocity acquired in one unit of time. Then at is the velocity acquired in the whole time t. And a body moving with this velocity for the time t would describe the space at^2 by Prop. 1. But a body moving from rest by a uniform force describes the space $\frac{1}{2}at^2$ by Prop. 5. Therefore the latter space is half the former. Q. E. D.

Cor. 1. A body falling from rest by the uniform force of gravity, describes 16 feet in one second.

Therefore with the velocity acquired it would describe 32 feet in one second. Therefore gravity generates a velocity of 32 feet in one second of time.

COR. 2. If g represent 32 feet, the space through which a body falls in t seconds by the action of gravity is $\frac{1}{2}gt^2$.

PROP. VII. When a body is projected in a direction opposite to the direction of a uniform force, with a velocity v, the whole time (t) of its motion till its velocity is destroyed, and the space (s) described in that time, are known by the equations $v = at$, $s = \frac{1}{2}at^2$.

For by the Definition of uniform force, the force, acting in a direction opposite to the motion, subtracts in equal times the same velocities which the same force adds when it acts in the direction of the motion. Therefore at a series of units of time the velocities will be v, $v - a$, $v - 2a$, $v - ta$, till $v - ta$ becomes 0, when all the velocity is destroyed; and when this occurs, $v - ta = 0$, or $v = ta$.

Also by Ax. 2, the same reasoning would hold as in Prop. 5, putting less for greater and greater for less; and therefore the same conclusion is true, namely, $s = \frac{1}{2}at^2$.

PROP. VIII. *Inductive Principle III. Second Law of Motion.* When any force acts upon a body in motion, the motion which the force would produce in the body at rest is compounded with the previous motion of the body.

The facts which this Induction includes are, in the first place, such as the following:—

(1). A stone dropped by a person in motion, is soon left behind.

From (1) it was inferred that if the earth were in motion, bodies dropt or thrown would be left behind.

But it appeared that the stone was not left behind so long as it was moving in free space, and was only stopt when it came to the ground. Again, it was found by experiment,

(2). That a stone dropt by a person in motion describes such a path that, relatively to him, it falls vertically.

(3). A man throwing objects and catching them again uses the same effort whether he be at rest or in motion.

Again, such facts as the following were considered:

(4). A stone thrown horizontally or obliquely describes a bent path and comes to the ground.

It was at first supposed that the stone does not fall to the ground till the original velocity is expended. But when the First Law of Motion was understood, it was seen that the gravity of the stone must, from the first, produce a change in the motion, and deflect the stone from the line in which it was thrown. And by more exact examination it appeared that (making allowance for the resistance of the air),—(5) the stone falls below the line of projection by exactly the space through which gravity in the same time would draw it from rest.

These facts were distinctly explained and rigorously deduced by introducing the *Definition* of *Composition of Motions*;—that two motions are compounded

when each produces its separate effect in a direction parallel to itself;

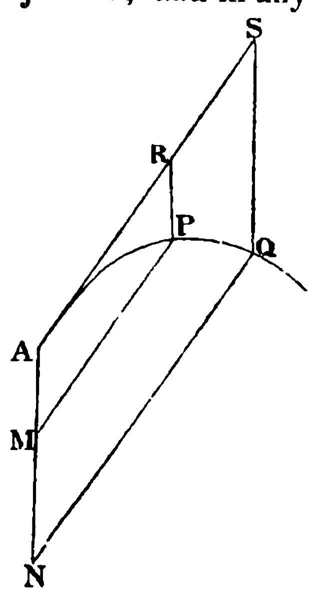
And the *Principle*, that when a force acts upon a body in motion, the motion which the force would produce in the body at rest is compounded with the previous motion of the body.

The Proposition is confirmed by shewing that its results, deduced by demonstration, agree with the facts.

DEF. 5. Two *motions* are *compounded* when each produces its separate effect in a direction parallel to itself.

PROP. IX. If a body be projected in any direction and acted upon by gravity, in any time it will describe a curve line of which, the tangent intercepted by the vertical line, and the vertical distance from the tangent, are respectively the spaces due to the original velocity and to the action of gravity in that time.

Let AT be the direction of projection; and in any time, let AR be the space which the body would have described with the velocity of projection in that time, and AM the space through which the body would have fallen in the same time. Then, completing the parallelogram $AMPR$, the body will, by the Second Law of Motion (Prop. 8) be found at P, and RP is equal to AM. Also AR is a tangent to the curve at A, because at A the body is moving in the direction AR. Therefore, &c. Q. E. D.



Cor. If P, Q be the points at which the projectile is found, at any two times t, T from its being at A, and if PR, QS be vertical lines, meeting the tangent at A in R, S, then

$$PR : QS :: AR^2 : AS^2.$$

For $PR : QS :: t^2 : T^2$ by Cor. to Prop. 5.

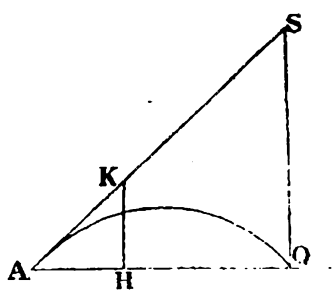
But $t : T :: AR : AS$; whence

$$t^2 : T^2 :: AR^2 : AS^2.$$

Therefore $PR : QS :: AR^2 : AS^2$.

Prop. X. A body is projected from a given point in a given direction; to find the range upon a horizontal plane, and the time of flight.

The *range* is the distance from the point of projection to the point where the *projectile* (or body projected) again strikes a plane passing through the point of projection.



Let a body be projected in a direction AK, such that AH, HK being horizontal and vertical, $AH : HK :: m : n$. Hence

$$\frac{AH}{HK} = \frac{m}{n}, \quad \frac{AK^2}{HK^2} = 1 + \frac{AH^2}{HK^2} = 1 + \frac{m^2}{n^2} = \frac{n^2 + m^2}{n^2};$$

$$\frac{HK}{AK} = \frac{n}{\sqrt{n^2 + m^2}}; \quad \frac{AH}{AK} = \frac{m}{n}\frac{HK}{AK} = \frac{m}{\sqrt{n^2 + m^2}}.$$

Let v be the velocity of projection, AQ the path described, QS vertical; and let the time of describing AQ be t. Therefore, by the last Proposition AS, the space described with velocity v in the time t, will be vt. Also SQ, the space fallen by gravity in the time t, will be $\frac{1}{2}gt^2$, by Prop. 6, Cor. 2.

And $\frac{SQ}{AS} = \frac{HK}{AK}$; that is, $\frac{\frac{1}{2}gt^2}{vt} = \frac{n}{\sqrt{n^2+m^2}}$,

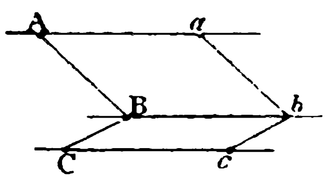
$\frac{gt}{2v} = \frac{n}{\sqrt{n^2+m^2}}$, $t = \frac{2v}{g} \cdot \frac{n}{\sqrt{m^2+n^2}}$; which is the time of flight.

Also $\frac{AQ}{AS} = \frac{AH}{AK}$, or $\frac{AQ}{vt} = \frac{m}{\sqrt{m^2+n^2}}$, $AQ =$

$vt\,\frac{m}{\sqrt{m^2+n^2}}$; $AQ = \frac{2v^2}{g}\,\frac{mn}{m^2+n^2}$, which is the range.

PROP. XI. If any particles, moving in parallel directions, and acted upon each by a certain force in the direction of its motion, move with velocities which are equal for all the particles at every instant, the motions of the particles will be the same if we suppose them to be connected so as to form a single rigid body, and the forces to be added together so as to form a single force.

Let A, B, C, be any particles acted upon by any forces, and moving in parallel directions with velocities which are equal at every instant. Since the velocities at every instant are equal, the spaces described in the same time are equal for all the particles, by Axiom 3.



Let Aa, Bb, Cc be the spaces described in any time, which are therefore equal and parallel. Therefore ab is equal and parallel to AB, and bc to BC, and so on. Therefore the relative positions and dis-

tances of the particles A, B, C are not altered by their motion into the places a, b, c.

Therefore, by Axiom 4, if we suppose the particles A, B, C to be rigidly connected, their motions will not be altered; that is, the motions will not be altered if A, B, C are supposed to be particles of a single rigid body.

Also, by Axiom 5, if we suppose the forces which act upon the particles A, B, C, to be added together so as to form a single force, the motion will not be altered.

Therefore, &c. Q. E. D.

PROP. XII. **If, on two bodies, two pressures act, which are proportional to the quantities of matter in the two bodies, the velocities produced in equal times in the two bodies are equal.**

Let P, Q, be two pressures, and m, n two bodies, measured by the number of units of quantity of matter which each contains; and let $P : Q :: m : n$.

Let the pressure P be divided into m parts, each of which will be $\frac{P}{m}$, and let each of these parts of the force act upon a separate one of the m units into which the body m can be divided, and let it produce in a time t a velocity v. Each of the pressures $\frac{P}{m}$ will produce in the unit upon which it acts for the time t, an equal velocity v, in a direction parallel to P. Therefore, if all the m pressures act for the same time t upon the m units of the body respectively, all the units will move with velocities which are equal at every instant. Therefore, by Prop. 11, if we suppose the m units to be connected so as to form one rigid

body m, and the forces to be added so as to form a single force P, the motion will still be the same. That is, the pressure P acting upon the body m, will produce the velocity v in the time t.

In the same manner it may be shewn that the pressure Q acting upon the body n will produce the same velocity which a pressure $\frac{Q}{n}$ produces in a body 1. But since $P : Q :: m : n$, $\frac{Q}{n} = \frac{P}{m}$; therefore $\frac{Q}{n}$ acting upon a body 1 will produce a velocity v in a time t. Therefore Q acting on n will produce a velocity v in a time t; the same which P produces in m. Q.E.D.

PROP. XIII. *INDUCTIVE PRINCIPLE IV. The Third Law of Motion.* When pressure generates (or destroys) motion in a given body the accelerating force is as the pressure.

The facts included in this Induction are such as the following:—

(1). When pressure produces motion, the velocity produced is greater when the pressure is greater.

In order to determine in what proportion the velocity increases with the pressure, further consideration and inquiry are necessary.

It appeared that,

(2). On an inclined plane the velocity acquired by falling down the plane is the same as that acquired by falling freely down the vertical height of the plane (Galileo's experiment).

(3). When two bodies P, Q* hang over a fixed pully, the heavier P descends, and the velocity gener-

* See figure to Prop. 17.

ated in a given time is as $P - Q$ directly, and as $P + Q$ inversely (Atwood's Machine).

(4). The small oscillations of pendulums are performed in times which are as the square roots of the lengths of the pendulums.

(5). In the impacts of bodies the momentum gained by the one body is equal to the momentum lost by the other (Newton's Experiments).

(6). In the mutual attractions of bodies the center of gravity remains at rest.

These results are distinctly explained and rigorously deduced by introducing the *Definition* of uniform Accelerating Force;—that it is as the velocity generated (or destroyed) in a given time;

And the *Principle* that the Accelerating Force for a given body is as the pressure.

Most of these consequences will be proved in the succeeding Propositions, (14, 15, 16, 17, 18), and thus this Inductive Proposition is confirmed.

DEF. 6. Uniform Accelerating Force is *measured* by the velocity generated in a unit of matter in a unit of time.

Hence in the formula in Prop. 4 and 5, a represents the Accelerating Force.

AXIOM 8. If two bodies move so that their velocities at every instant are equal, the Accelerating Forces of the two bodies at every instant are equal; and conversely.

AXIOM 9. If two bodies move so that their Accelerating Forces at every instant are in a constant ratio, and are in the direction of the motion, the velocities added or subtracted in any time are in the ratio of the Accelerating Forces.

Prop. XIV. **In different bodies, the Accelerating Force is as the pressure which produces motion directly, and as the quantity of matter moved inversely.**

Let two pressures P, Q, produce motion in two bodies of which the quantities of matter are M, N. Let $M : N :: P : X$; therefore, by Prop. 12, the force X would, in a given time, produce in N the same velocity which P would produce in M; that is, the Accelerating Force on M arising from the pressure P, is equal to the Accelerating Force on N arising from the pressure X.

But by the Third Law of Motion (Prop. 13) the Accelerating force on N arising from the pressure X is to the Accelerating Force on the same body N arising from the pressure Q as X is to Q.

Therefore, the Accelerating Force on M arising from P is to the Accelerating Force on N arising from Q as X is to Q.

But $M : N :: P : X$; therefore $X = \dfrac{PN}{M}$, and therefore X is to Q, as $\dfrac{PN}{M}$ is to Q, or as $\dfrac{P}{M}$ to $\dfrac{Q}{N}$.

Therefore the Accelerating Forces of P on M and of Q on N are as $\dfrac{P}{M}$ and $\dfrac{Q}{N}$. Q. E. D.

Prop. XV. **On the inclined plane, the time of falling down the plane is to the time of falling freely down the vertical height of the plane as the length of the plane to its height.**

Let L be the length of the plane, H its height. The pressure which urges a body down an inclined

plane is equal to the pressure which would support it acting in the opposite direction; but this pressure : W the weight of the body :: H : L (B. I. Prop. 20.) Therefore the pressure which produces motion on the plane is $\frac{WH}{L}$.

The quantity of matter of the body is as W.

Hence, since by the last Proposition the Accelerating Force on the inclined plane is as the pressure directly and the quantity of matter inversely; therefore Accelerating Force on Inclined Plane : Accelerating Force of body falling freely :: $\frac{WH}{WL} : \frac{W}{W}$ :: $H : L$.

Now the force on the inclined plane is a uniform accelerating force; and therefore the velocity acquired in a unit of time measures it, by Def. 6. Therefore, if La be the velocity acquired in a unit of time by a body falling freely, Ha will be the velocity acquired in a unit of time down the inclined plane. And the rule of Prop. 5 is applicable. Therefore, if t and t' be the times of falling down L, and of falling vertically down H,

$$L : H :: \tfrac{1}{2}Hat^2 : \tfrac{1}{2}Lat'^2;$$
$$\text{or } L^2 : H^2 :: t^2 : t'^2;$$
$$\text{or } L : H :: t : t'.$$

Prop. XVI. **On the inclined plane, the velocity acquired by falling down the inclined plane is equal to the velocity acquired by falling freely down the vertical height of the plane.**

As before, the accelerating force on the plane is to the accelerating force of a body falling freely

$$:: H : L;$$

Also $s = \frac{1}{2} vt$, by Prop. 6; whence as before, v' being the velocity acquired by falling freely down H,

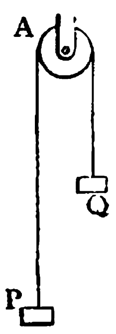
$$L : H :: \tfrac{1}{2}vt : \tfrac{1}{2}v't' :: vt : v't';$$

But $H : L \quad :: t' : t$ by Prop. 15.

Therefore $1 : 1 \quad :: v : v'$;

Whence $v = v'$; the velocity acquired down the plane is equal to the velocity acquired down the vertical height. Q. E. D.

Prop. XVII. When two bodies P, Q hang over a fixed pully, and move by their own weight merely,* the heavier P descends, and the lighter Q ascends, by the action of an accelerating force which is as $\dfrac{P-Q}{P+Q}$.



The string which connects P and Q exerts an equal pressure in opposite directions upon P and upon Q, (Axiom 6). Let this pressure be X. Then since P is urged downwards by a force P and upwards by a force X, it is on the whole urged downwards by a force $P - X$. And the quantity of matter is P. Therefore, by Prop. 14, the Accelerating Force upon P downwards is as $\dfrac{P-X}{P}$. In the same manner, since X acts upwards upon Q and the weight of Q acts downwards, the accelerating force upon Q upwards is as $\dfrac{Q-X}{Q}$. But the accelerating force upon Q upwards and upon P downwards must be equal, because they move at every point with equal velocities, by Axiom 8.

* That is, neglecting the effect of the matter in the pully and the string.

Therefore $\frac{X - Q}{Q}$ is equal to $\frac{P - X}{P}$;

that is, $\frac{X}{Q} - 1$ is equal to $1 - \frac{X}{P}$;

or $\frac{X}{Q} + \frac{X}{P}$ is equal to 2.

Therefore $\frac{X(P + Q)}{PQ}$ is equal to 2;

and X is equal to $\frac{2PQ}{P + Q}$.

Hence $P - X$ is $P - \frac{2PQ}{P + Q}$, or $\frac{P^2 - PQ}{P + Q}$; and the accelerating force upon P, which is as $\frac{P - X}{P}$, is as $\frac{P - Q}{P + Q}$. And, in like manner, the accelerating force upon Q is as $\frac{P - Q}{P + Q}$.

DEF. 7. The *momentum* of a body is the product of the numbers which express its velocity and its quantity of matter.

DEF. 8. *Elastic* bodies are those which separate when one impinges upon another; *inelastic* bodies are those which do not separate after impact.

DEF. 9. The impact of two bodies is *direct*, when the bodies, before impact, either moving in the same direction or one of them being at rest, the pressure which each exerts upon the other is in the direction of the motion.

PROP. XVIII. In the direct impact of two bodies the momenta gained and lost are equal.

Let P impinge upon Q directly, and let X be the pressure which each exerts upon the other at any instant. Therefore the accelerating forces which act upon P and Q in opposite directions are as $\frac{X}{P}$ and $\frac{X}{Q}$; and are therefore at every instant in the constant ratio of $\frac{1}{P}$ to $\frac{1}{Q}$, or of Q to P. Therefore, by Ax. 9, the velocities generated in Q and destroyed in P, in any time, are in the same constant ratio of Q to P. And the quantities of matter are as P and Q. Therefore, by Def. 7, the momentum generated in Q and the momentum destroyed in P, in any time, are as PQ to PQ; that is, they are equal. Q. E. D.

Cor. 1. If P and Q are elastic, they will separate after the impact; and the momenta generated and destroyed in Q and P by the elasticity will still be equal, for the same reasons as before.

Cor. 2. The velocity destroyed in P, according to Cor. 1, may be greater than its whole velocity. In this case, P will, after the impact, move in the opposite direction with a velocity which is the excess of the velocity lost over the original velocity.

Cor. 3. Before the impact, Q may move in a direction opposite to P. In this case the velocity gained by Q is to be understood as including the velocity in the opposite direction, which is destroyed.

Cor. 4. If two bodies P and Q, move in opposite directions with velocities which are in the ratio of Q to P, they will be at rest after impact if they are inelastic. For since they are inelastic, they will not

separate after impact: therefore they will either be at rest or move on together. But if they move in the direction of P's motion, P has lost less than its whole velocity, and Q has gained more than its own velocity. But this is impossible, for the velocities lost and gained are in the ratio of Q to P; that is, in the ratio of P's velocity to Q's velocity. Therefore the bodies do not move in the direction of P's motion. And, in like manner, it may be shown that they do not move in the direction of Q's motion. Therefore they remain at rest.

PROP. XIX. The mutual pressure, attraction, or repulsion, or direct impact of two bodies, cannot put in motion their centre of gravity.

Let two bodies P, Q, act upon each other by pressure, attraction, or repulsion, the force which each exerts upon the other (Axiom 6) being X. Therefore (Prop. 14) the accelerating forces which act on P and Q are as $\frac{X}{P}$ and $\frac{X}{Q}$ respectively, or in the constant ratio of Q to P. Therefore the velocities acquired by P and Q in any equal times are in this ratio by Axiom 9, and therefore the spaces are in the same ratio by Axiom 3.

Let P, Q, be any two bodies of which the center of gravity is C, which is at first at rest. Therefore

by B. 1, Prop. 24, $Q : P :: CP : CQ$, and $CQ = \frac{P}{Q} CP$. And if Pp, Qq be any spaces described in equal times, by the mutual pressure, attraction, or repulsion of the bodies, it has been proved that $Q : P$

:: $Pp : Qq$; and therefore $Qq = \frac{P}{Q} Pp$. Hence, subtracting, it follows that $Cq = \frac{P}{Q} Cp$, or $Q : P :: Cp : Cq$. And therefore C is still the center of gravity of the bodies P, Q, when they are come into the positions p, q; that is, the center of gravity has not been put in motion.

Also if the two bodies P, Q, not attracting or repelling each other, move towards each other with uniform velocities which are in the ratio of Q to P, and impinge; the spaces described in any time (as Pp, Qq) will be in the same ratio of Q to P, and, as above, the center of gravity will be at rest. And when the bodies impinge on each other, the velocities of each will either be destroyed, or destroyed and generated in an opposite direction; and in either case, since the mutual pressure is equal on both, the accelerating forces which destroy and generate velocity, will be in the ratio of Q to P, as in Prop. 17. Therefore the velocities destroyed and generated are in the same ratio as the original velocities. Therefore if the whole velocity of one body is destroyed, the whole velocity of the other body also is destroyed, and the bodies are both at rest, and their center of gravity is still at rest after impact.

But if the velocities be destroyed, and velocities generated in an opposite direction, these new velocities will also be in the ratio of the original velocities, because the accelerating forces at every instant are so, (Ax. 9); and therefore the spaces described in any time by the new velocities will be in the same ratio; and therefore, as before, it may be shown that C is still the center of gravity of P, Q.

Therefore, under all the circumstances stated, the center of gravity remains at rest. Q. E. D.

Examples to Propositions 4, 5, 6, 7, 10, 17, 18.

By means of these Propositions, we can solve such Examples as the following:—

When a body falls freely by the action of gravity, the quantity a in Prop. 4 is 32 feet, the unit of time being one second, and $v = gt$. Also (Prop. 6, Cor. 2) $s = \frac{1}{2} gt^2$.

Ex. 1. To find the velocity acquired by a body which falls by gravity for 30 seconds.

$$v = gt = 32 \times 30 = 960 \text{ feet per second.}$$

2. To find the space fallen through in the same time, $s = \frac{1}{2} gt^2 = 16 \times 30^2 = 14400$ feet.

3. To find in what time a body falls through 1024 feet.

$$1024 = 16 \times t^2, \quad t^2 = 64, \quad t = 8 \text{ seconds.}$$

4. To find the velocity acquired in the same space, $v = gt = 32 \times 8 = 256$ feet per second.

5. A body is projected directly upwards, with a velocity of 1000 feet a second; how high will it go?

By Prop. 7, the height will be that through which a body must fall to acquire the same velocity.

Now since

$$v = gt, \; 1000 = 32\,t, \; t = \frac{1000}{32} = \frac{125}{4} = 31\tfrac{1}{4}''.$$

$$s = \frac{1}{2} gt^2 = 16\,\frac{(125)^2}{4^2} = (125)^2 = 15625 \text{ feet.}$$

6. A body is projected with a velocity of 32 feet a second in a direction which makes with the horizon half a right angle: to find the time of flight and the range.

In this case $m = n$; therefore, by Prop. 10,

$$\frac{n}{\sqrt{n^2 + m^2}} = \frac{1}{\sqrt{2}}; \; t = \frac{2v}{g} \cdot \frac{1}{\sqrt{2}} = \frac{2 \times 32}{32 \times \sqrt{2}}$$

$$= \sqrt{2} = 1 \,.\, 4 \text{ seconds};$$

$$\text{the range} = \frac{2v^2}{g} \cdot \frac{mn}{m^2 + n^2} = \frac{2 \times (32)^2}{32} \frac{1}{2} = 32 \text{ feet.}$$

7. A cannon ball is projected with a velocity of 1600 feet a second, in a direction which rises 3 feet in 4 feet horizontal: find the time of flight and the range

$$\frac{n}{\sqrt{n^2 + m^2}} = \frac{3}{\sqrt{9 + 16}} = \frac{3}{5}; \; t = \frac{2 \times 1600}{32} \times \frac{3}{5}$$

$$= 75 \text{ seconds};$$

$$\text{the range} = \frac{2v^2}{g} \frac{nm}{m^2 + n^2} = \frac{2\,(1600)^2}{32} \times \frac{12}{25}$$

$$= 76800 \text{ feet.}$$

8. An inelastic body A impinges directly on another inelastic body B at rest, with a velocity of 10 feet a second; A being 3 and B 2 ounces, find the velocity after impact.

If x be the velocity of both after impact, the velocity lost by A is $10 - x$, and the velocity gained by B is x. Hence the momentum lost by A is $3 \times (10 - x)$; and that gained by B is $2 \times x$: and these are equal by Prop. 18; therefore

$$3\,(10 - x) = 2x, \;\; 30 = 3x + 2x, \;\; x = 6.$$

9. The bodies being perfectly elastic, find the motions after impact.

In perfectly elastic bodies, the velocity lost by A and the velocity gained by B in the restitution of the figure are equal to the velocity lost by A and gained by B in the compression.

Now the velocity lost by *A* in the compression is $10 - 6$ or 4; therefore the whole velocity lost by *A* is 8, and its remaining velocity 2.

And the velocity gained by *B* in the compression is 6, and therefore the whole velocity gained by *B* is 12, which is *B*'s velocity after impact.

10. A body *A* (3 ounces) draws *B* (2 ounces) over a fixed pully: find the space described in one second from rest.

By Prop. 17, the accelerating force is as $\frac{3 - 2}{3 + 2}$; that is, it is $\frac{1}{5}$ of gravity; and the space in a second is as the force: therefore the space described in one second is $\frac{16}{5}$, or $3\frac{1}{5}$ feet.

REMARKS

ON

MATHEMATICAL REASONING,

AND ON

THE LOGIC OF INDUCTION.

SECT. I. *On the Grounds of Mathematical Reasoning.*

1. THE study of a science, treated according to a rigorous system of mathematical reasoning, is useful, not only on account of the positive knowledge which may be acquired on the subjects which belong to the science, but also on account of the collateral effects and general bearings of such a study, as a discipline of the mind and an illustration of philosophical principles.

Considering the study of the mathematical sciences with reference to these latter objects, we may note two ways in which it may promote them;—by habituating the mind to strict reasoning,—and by affording an occasion of contemplating some of the most important mental processes and some of the most distinct forms of truth. Thus mathematical studies may be useful in teaching practical logic and theoretical metaphysics. We shall make a few remarks on each of these topics.

2. The study of Mathematics teaches strict reasoning—by bringing under the student's notice prominent and clear examples of trains of demonstration:—by exercising him in the habits of attentive and connected thought which are requisite in order to follow these trains;—and by familiarizing him with the peculiar and distinctive conviction which demonstration produces, and with the rigorous exclusion of all considerations which do not enter into the demonstration.

3. Logic is a system of doctrine which lays down rules for determining in what cases pretended reasonings are and are not demonstrative. And accordingly, the teaching of strict reasoning by means of the study of logic is often recommended and practised. But in order to shew the superiority of the study of mathematics for this purpose, we may consider,—that reasoning, as a practical process, must be learnt by practice, in the same manner as any other practical art, for example, riding, or fencing;—that we are not secured from committing fallacies by such a classification of fallacies as logic supplies, as a rider would not be secured from falls by a classification of them;—and that the habit of attending to our mental processes while we are reasoning, rather interferes with than assists our reasoning well, as the horseman would ride worse rather than better, if he were to fix his attention upon his muscles when he is using them.

4. To this it may be added, that the peculiar habits which enable any one to follow a *chain* of reasoning are excellently taught by mathematical study, and are hardly at all taught by logic. These habits consist in not only apprehending distinctly the demonstration of a proposition when it is proved, but in

retaining all the propositions thus proved, and using them in the ulterior steps of the argument with the same clear conviction, readiness, and familiarity, as if they were self-evident principles. Writers on Logic seldom give examples of reasoning in which several syllogisms follow each other; and they never give examples in which this progressive reasoning is so exemplified as to make the process familiar. Their chains generally consist only of two or three links. In Mathematics, on the contrary, every theorem is an example of such a chain; every proof consists of a series of assertions, of which each depends on the preceding, but of which the last inferences are no less evident or less easily applied than the simplest first principles. The language contains a constant succession of short and rapid references to what has been proved already; and it is justly assumed that each of these brief movements helps the reasoner forwards in a course of infallible certainty and security. Each of these hasty glances must possess the clearness of intuitive evidence, and the certainty of mature reflection; and yet must leave the reasoner's mind entirely free to turn instantly to the next point of his progress. The faculty of performing such mental processes well and readily is of great value, and is in no way fostered by the study of logic.

5. It is sometimes objected to the study of Mathematics as a discipline of reasoning, that it tends to render men insensible to all reasoning which is not mathematical, and leads them to demand, in other subjects, proofs such as the subject does not admit of, or such as are not appropriate to the matter.

To this it may be replied, that these evil results, so far as they occur, arise either from the student

pursuing too exclusively one particular line of mathematical study, or from erroneous notions of the nature of demonstration.

The present volume is intended to assist, in some measure, in remedying the too exclusive pursuit of one particular line of Mathematics, by shewing that the same simplicity and evidence which are seen in the Elements of Geometry may be introduced into the treatment of another subject of a kind very different; and it is hoped that we may thus bring this subject within the reach of those who cultivate the study of Mathematics as a discipline only. The remarks now offered to the reader are intended to aid him in forming a just judgment of the analogy between mathematical and other proof; which is to be done by pointing out the true grounds of the evidence of Geometry, and by exhibiting the views which are suggested by the extension of mathematical reasoning to sciences concerned about physical facts.

6. We shall therefore now proceed to make some remarks on the nature and principles of reasoning, especially as far as they are illustrated by the mathematical sciences.

Some of the leading principles which bear upon this subject are brought into view by the consideration of the question, "What is the foundation of the certainty arising from mathematical demonstration?" and in this question it is implied that mathematical demonstration is recognised as a kind of reasoning possessing a peculiar character and evidence, which make it a definite and instructive subject of consideration.

7. Perhaps the most obvious answer to the question respecting the conclusiveness of mathematical demonstration is this;—that the certainty of such demon-

stration arises from its being founded upon *Axioms*; and conducted by steps, of which each might, if required, be stated as a rigorous *Syllogism*.

This answer might give rise to the further questions, What is the foundation of the conclusiveness of a Syllogism? and, What is the foundation of the certainty of an Axiom? And if we suppose the former enquiry to be left to Logic, as being the subject of that science, the latter question still remains to be considered. We may also remark upon this answer, that mathematical demonstration appears to depend upon Definitions, at least as much as upon Axioms. And thus we are led to these questions:—Whether mathematical demonstration is founded upon Definitions, or upon Axioms, or upon both? and, What is the real nature of Definitions and of Axioms?

8. The question, What is the foundation of mathematical demonstration? was discussed at considerable length by Dugald Stewart*; and the opinion at which he arrived was, that the certainty of mathematical reasoning arises from its depending upon *definitions*. He expresses this further, by declaring that mathematical truth is hypothetical, and must be understood as asserting only, that *if* the definitions are assumed, the conclusion follows. The same opinion has, I think, prevailed widely among other modern speculators on the same subject, especially among mathematicians themselves.

9. In opposition to this opinion, I urge, in the first place, that no one has yet been able to construct a system of mathematical truth by means of definitions alone, to the exclusion of axioms; although attempts having this tendency have been made constantly and

* Elements of the Philosophy of the Human Mind, Vol. II.

earnestly. It is, for instance, well known to most readers, that many mathematicians have endeavoured to get rid of Euclid's "Axioms" respecting straight lines and parallel lines; but that none of these essays has been generally considered satisfactory. If these axioms could be superseded, by definition or otherwise, it was conceived that the whole structure of Elementary Geometry would rest merely upon definitions; and it was held by those who made such essays, that this would render the science more pure, simple, and homogeneous. If these attempts had succeeded, Stewart's doctrine might have required a further consideration; but it appears strange to assert that Geometry is supported by definitions, and not by axioms, when she cannot stir four steps without resting her foot upon an axiom.

10. But let us consider further the nature of these attempts to supersede the axioms above mentioned. They have usually consisted in endeavours so to frame the definitions, that these might hold the place which the axioms hold in Euclid's reasoning. Thus the axiom, that "two straight lines cannot enclose a space," would be superfluous, if we were to take the following definition:—"A line is said to be *straight*, when two such lines cannot coincide in *two* points without coinciding *altogether*."

But when such a method of treating the subject is proposed, we are unavoidably led to ask,—whether it is allowable to lay down such a definition. It cannot be maintained that we may propound any form of words whatever as a definition, without any consideration whether or not it suggests to the mind any intelligible or possible conception. What would be said, for instance, if we were to state the following as a definition, "A line is said to be *straight* (or any other

term) when two such lines cannot coincide in *one* point without coinciding altogether?" It would inevitably be remarked, that no such lines exist; or that such a property of lines cannot hold good without other conditions than those which this definition expresses; or, more generally, that the definition does not correspond to any conception which we can call up in our minds, and therefore can be of no use in our reasonings. And thus it would appear, that a definition, to be admissible, must necessarily refer to and agree with some conception which we can distinctly frame in our thoughts.

11. This is obvious, also, by considering that the definition of a straight line could not be of any use, except we were entitled to apply it in the cases to which our geometrical propositions refer. No definition of straight lines could be employed in Geometry, unless it were in some way certain that the lines so defined are those by which angles are contained, those by which triangles are bounded, those of which parallelism may be predicated, and the like.

12. The same necessity for some general conception of such lines accompanying the definition, is implied in the terms of the definition above suggested. For what is there meant by "*such* lines?" Apparently, lines having some general character in which the property is necessarily involved. But how does it appear that lines may have such a character? And if it be self-evident that there may be such lines, this evidence is a necessary condition of this (or any equivalent) definition. And since this self-evident truth is the ground on which the course of reasoning must proceed, the simple and obvious method is, to state the property *as* a self-evident truth; that is, as an axiom.

Similar remarks would apply to the other axiom above mentioned; and to any others which could be proposed on any subject of rigorous demonstration.

13. If it be conceded that such a conception accompanying the definition is necessary to justify it, we shall have made a step in our investigation of the grounds of mathematical evidence. But such an admission does not appear to be commonly contemplated by those who maintain that the conclusiveness of mathematical proof results from its depending on definitions. They generally appear to understand their tenet as if it implied *arbitrary* definitions. And something like this seems to be held by Stewart, when he says that mathematical truths are true *hypothetically*. For we understand by an hypothesis a supposition, not only which we may make, but may abstain from making, or may replace by a different supposition.

14. That the fundamental conceptions of Geometry are not arbitrary definitions, or selected hypotheses, will, I think, be clear to any one who reasons geometrically at all. It is impossible to follow the steps of any single proposition of Geometry without conceiving a straight line and its properties, whether or not such a line be defined, and whether or not its properties be stated. That a straight line should be distinguished from all other lines, and that the axiom respecting it should be seen to be true, are circumstances indispensable to any clear thought on the subject of lines. Nor would it be possible to frame any coherent scheme of Geometry in which straight lines should be excluded, or their properties changed. Any one who should make the attempt, would betray, in his first propositions, to all men who can reason geometrically, a reference to straight lines.

15. If, therefore, we say that Geometry depends on definitions, we must add, that they are *necessary*, not arbitrary definitions,—such definitions as we must have in our minds, so far as we have elements of reasoning at all. And the elementary hypotheses of Geometry, if they are to be so termed, are not hypotheses which are requisite to enable us to reach this or that conclusion; but hypotheses which are requisite for *any* exercise of our thoughts on such subjects.

16. Before I notice the bearing of this remark on the question of the necessity of axioms, I may observe that Stewart's disposition to consider definitions, and not axioms, as the true foundation of Geometry, appears to have resulted, in part, from an arbitrary selection of certain axioms, as specimens of all. He takes, as his examples, the axioms, "that if equals be added to equals the wholes are equal," that "the whole is greater than its part;" and the like. If he had, instead of these, considered the more properly geometrical axioms,—such as those which I have mentioned; "that two straight lines cannot enclose a space;" or any of the axioms which have been made the basis of the doctrine of parallels; for instance, Playfair's axiom, "that two straight lines which intersect each other cannot both of them be parallel to a third straight line;"—it would have been impossible for him to have considered axioms as holding a different place from definitions in geometrical reasoning. For the properties of triangles are proved from the axiom respecting straight lines, as distinctly and directly, as the properties of angles are proved from the definition of a right angle. Of the many attempts made to prove the doctrine of parallels, almost all professedly, all really, assume some axiom or axioms which are the basis of the reasoning.

17. It is therefore very surprising that Stewart should so exclusively have fixed his attention upon the more general axioms, as to assert, following Locke, "that from [mathematical] axioms it is not possible for human ingenuity to draw a single inference*;" and even to make this the ground of a contrast between geometrical axioms and definitions. The slightest examination of any treatise of Geometry might have shown him that there is no sense in which this can be asserted of axioms, in which it is not equally true of definitions; or rather, that while Euclid's definition of a straight line leads to no truth whatever, his axiom respecting straight lines is the foundation of the whole of Geometry; and that, though we can draw some inferences from the definition of parallel straight lines, we strive in vain to complete the geometrical doctrine of such lines, without assuming some axiom which enables us to prove the converse of our first propositions. Thus, that which Stewart proposes as the distinctive character of axioms, fails altogether; and with it, as I conceive, the whole of his doctrine respecting mathematical evidence.

18. That Geometry (and other sciences when treated in a method equally rigorous) depends upon axioms as well as definitions, is supposed by the form in which it is commonly presented. And after what we have said, we shall assume this form to be a just representation of the real foundations of such sciences, till we can find a tenable distinction between axioms and definitions, in their nature, and in their use; and till we have before us a satisfactory system of Geometry without axioms. And this system, we may remark, ought to include the Higher as well as the

* Elements of the Philosophy of the Human Mind, Vol. II. p. 38.

Elementary Geometry, before it can be held to prove that axioms are needless; for it will hardly be maintained, that the properties of circles depend upon definitions and hypotheses only, while those of ellipses require some additional foundation: or that the comparison of curve lines requires axioms, while the relations of straight lines are independent of such principles.

19. Having then, I trust, cleared away the assertion, that mathematical reasoning rests ultimately upon definitions only, and that this is the ground of its peculiar cogency, I have to examine the real evidence of the truth of such axioms as are employed in the exact Mathematical Sciences. And we are, I think, already brought within view of the answer to this question. For if the definitions of Mathematics are not arbitrary, but necessary, and must, in order to be applicable in reasoning, be accompanied by a conception of the mind through which this necessity is seen; it is clear that this apprehension of the necessity of the properties which we contemplate, is really the ground of our reasonings and the source of their irresistible evidence. And where we clearly apprehend such necessary relations, it can make no difference whatever in the nature of our reasoning, whether we express them by means of definitions or of axioms. We define a straight line vaguely;—that it is that line which lies evenly between two points: but we forthwith remedy this vagueness, by the axiom respecting straight lines: and thus we express our conception of a straight line, so far as is necessary for reasoning upon it. We might, in like manner, begin by defining a right angle to be the angle made by a line which stands evenly between the two portions of another line; and we might add an axiom, that all right angles are equal. Instead of

this, we define a right angle to be that which a line makes with another when the two angles on the two sides of it are equal. But in all these cases, we express our conception of a necessary relation of lines; and whether this be done in the form of definitions or axioms, is a matter of no importance.

20. But it may be asked, If it be thus unimportant whether we state our fundamental principles as axioms or definitions, why not reduce them all to definitions, and thus give to our system that aspect of independence which many would admire, and with which none need be displeased? And to this we answer, that if such a mode of treating the subject were attempted, our definitions would be so complex, and so obviously dependent on something not expressed, that they would be admired by none. We should have to put into each definition, as conditions, all the axioms which refer to the things defined. For instance, who would think it a gain to escape the difficulties of the doctrine of parallels by such a definition as this: "Parallel straight lines are those which being produced indefinitely both ways do not meet; and which are such that if a straight line intersects one of them it must somewhere meet the other?" And in other cases, the accumulation of necessary properties would be still more cumbersome and more manifestly heterogeneous.

21. The reason of this difficulty is, that our fundamental conceptions of lines and other relations of space, are capable of being contemplated under several various aspects, and more than one of these aspects are needed in our reasonings. We may take one such aspect of the conception for a definition; and then we must introduce the others by means of ax-

ioms. We may define parallels by their not meeting; but we must have some positive property, besides this negative one, in order to complete our reasonings respecting such lines. We have, in fact, our choice of several such self-evident properties, any of which we may employ for our purpose, as geometers well know; but with our naked definition, as they also know, we cannot proceed to the end. And in other cases, in like manner, our fundamental conception gives rise to various elementary truths, the connexion of which is the basis of our reasonings: but this connexion resides in our thoughts, and cannot be made to follow, as a logical result, from any assumed form of words, presented as a definition.

22. If it be further demanded, What is the nature of this bond in our thoughts by which various properties of lines are connected? perhaps the simplest answer is to say, that it resides in *the idea of space.* We cannot conceive things in space without being led to consider them as determined and related in some way or other to straight lines, right angles, and the like; and we cannot contemplate these determinations and relations distinctly, without assuming those properties of straight lines, of right angles, and of the rest, which are the basis of our Geometry. We cannot conceive or perceive objects at all, except as existing in space; we cannot contemplate them geometrically, without conceiving them in space which is subjected to geometrical conditions; and this mode of contemplation is, by language, analysed into definitions, axioms, or both.

23. The truths thus seen and known, may be said to be known by *intuition.* In English writers this term has, of late, been vaguely used, to express all

convictions which are arrived at without conscious reasoning, whether referring to relations among our perceptions, or to conceptions of the most derivative and complex nature. But if we were allowed to restrict the use of this term, we might conveniently confine it to those cases in which we necessarily apprehend relations of things truly, as soon as we conceive the objects distinctly. In this sense axioms may be said to be known *by intuition*; but this phraseology is not essential to our purpose.

24. It appears, then, that the evidence of the axioms of Geometry depends upon a distinct possession of the idea of space. These axioms are stated in the beginning of our Treatises, not as something which the reader is to learn, but as something which he already knows. No proof is offered of them; for they are the beginning not the end of demonstrations. The student's clear apprehension of the truth of these is a condition of the possibility of his pursuing the reasonings on which he is invited to enter.* Without this mental capacity, and the power of referring to it, in the reader, the writer's assertions and arguments are empty

* In this statement respecting the nature of Axioms, I find myself agreeing with the acute author of "Sematology." See the "Sequel to Sematology," p. 103. "An Axiom does not account for an intellection; it does but describe the requisite competency for it." It appears to me that this view is not familiar among English metaphysicians. I may here quote what I said at a former period, "However we may *define* force, it is necessary in order to understand the elementary reasonings of this portion of science, that we should *conceive* it distinctly. Do we wish for a test of the distinctness of our conceptions? The test is, our being able to see the necessary truth of the Axioms on which our reasonings rest...These principles (the Axioms of Statics) are all perfectly evident as soon as we have formed the general conception of pressure; but without that act of thought, they can have no evidence whatever given them by any form of words, or reference to other truths;—by definitions, or by illustrations from other kinds of quantity." *Thoughts on the Study of Mathematics*, p. 25.

and unmeaning words; but then, this capacity and power are what all rational creatures alike possess, though habit may have developed it in very various degrees in different persons.

25. It has been common in the school of metaphysicians of which I have spoken, to describe some of the elementary convictions of our minds as *fundamental laws* of belief; and it appears to have been considered that this might be taken as a final and sufficient account of such convictions. I do not know whether any persons would be tempted to apply this formula, as a solution of our question respecting the nature of axioms. If this were proposed, I should observe, that this form of expression seems to me, in such a case, highly unsatisfactory. For *laws* require and enjoin a conjunction of things which can be contemplated separately, and which would be disjoined if the law did not exist. It is a law of nature that terrestrial bodies, when free, fall downwards; for we can easily conceive such bodies divested of such a property. But we cannot say, in the same sense, that the impossibility of two straight lines inclosing a space arises from a law; for if they are straight lines, they need no law to compel this result. We cannot conceive straight lines exempt from such a law. To speak of this property as imposed by a law, is to convey an inadequate and erroneous notion of the close necessity, inviolable even in thought, by which the truth clings to the conception of the lines.

26. This expression, of "laws of belief," appears to have found favour, on this account among others, that it recognised a kind of analogy between the grounds of our reasoning on very abstract subjects, and the principles to which we have recourse in other cases when we manifestly derive our fundamental

truths from facts, and when it is supposed to be the ultimate and satisfactory account of them to say, that they are laws of nature learnt by observation. But such an analogy can hardly be considered as a real recommendation by the metaphysician; since it consists in taking a case in which our knowledge is obviously imperfect and its grounds obscure, and in erecting this case into an authority which shall direct the process and control the enquiry of a much more profound and penetrating kind of speculation. It cannot be doubted that we are likely to see the true grounds and evidence of our doctrines much more clearly in the case of Geometry and other rigorous systems of reasoning, than in collections of mere empirical knowledge, or of what is supposed to be such. It is both an unphilosophical and an indolent proceeding, to take the latter cases as a standard for the former.

27. I shall therefore consider it as established, that in Geometry our reasoning depends upon axioms as well as definitions,—that the evidence of the truth of the axioms and of the propriety of the definitions resides in the idea of space,—and that the distinct possession of this idea, and the consequent apprehension of the truth of the axioms which are its various aspects, is supposed in the student who is to pursue the path of geometrical reasoning. This being understood, I have little further to observe on the subject of Geometry. I will only remark—that all the conclusions which occur in the science follow purely from those first principles of which we have spoken;—that each proposition is rigorously proved from those which have been proved previously from such principles;—that this process of successive proof is termed *Deduction*;—and that the rules which se-

cure the rigorous conclusiveness of each step are the rules of *Logic*, which I need not here dwell upon.

28. But I now proceed to consider some other questions to which our examination of the evidence of Geometry was intended to be preparatory;—How far do the statements hitherto made apply to other sciences? for instance, to such sciences as are treated of in the present volume, Mechanics and Hydrostatics. To this I reply, that some such sciences at least, as for example Statics, appear to me to rest on foundations exactly similar to Geometry:—that is to say, that they depend upon axioms,—self-evident principles, not derived in any immediate manner from experiment, but involved in the very nature of the conceptions which we must possess, in order to reason upon such subjects at all. The proof of this doctrine must consist of several steps, which I shall take in order.

29. In the first place, I say that the axioms of Statics are *self-evidently true*. In the beginning of the preceding Treatise I have stated these barely as axioms, without addition or explanation, as the axioms of Geometry are stated in treatises on that subject. And such is the proper and orderly mode of exhibiting axioms; for, as has been said, they are to be understood as an expression of the condition of conception of the student. They are not to be learnt from without, but from within. They necessarily and immediately flow from the distinct possession of that idea, which if the student do not possess distinctly, all conclusive reasoning on the subject under notice is impossible. It is not the business of the deductive reasoner to communicate the apprehension of these truths, but to deduce others from them.

30. But though it may not be the author's business to elucidate the truth of the axioms as a deductive reasoner, it may still be desirable that he should do so as a philosophical teacher; and though it may not be possible to add anything to their evidence in the mind of him who possesses distinctly the idea from which they flow, it may be in our power to assist the beginner in obtaining distinct possession of this idea and unfolding it into its consequences. I shall therefore make a few remarks, tending to illustrate the self-evident nature of the "Axioms" of Statics, of Hydrostatics, and of the Doctrine of Motion.

31. Omitting, for the present, the consideration of the First Axiom of Statics (see p. 28); the Second is, "If two equal forces act perpendicularly at the extremities of equal arms of a straight line to turn it opposite ways, they will keep each other in equilibrium." This is often, and properly, further confirmed, by observing that there is no reason why one of the forces should preponderate rather than the other, and that, as both cannot preponderate, neither will do so. All the circumstances on which the result (equilibrium or preponderance) can depend, are equal on the two sides;—equal arms, equal angles, equal forces. If the forces are not in equilibrium, *which* will preponderate? no answer can be given, because there is no circumstance left by which either can be distinguished.

32. The argument which we have just used, is often applicable, and may be expressed by the formula, "there is no reason why one of the two opposite cases should occur, which is not equally valid for the other; and as both cannot occur (for they are opposite cases) neither will occur." This argument is called "the principle of sufficient reason;" it puts in a general

form the considerations on which several of our axioms depend; and to persons who are accustomed to such generality, it may make their truth more clear.

The same principle might be applied to other cases, for example to Axiom 7, that the effect produced on a bent lever does not depend on the direction of the arm. For if we suppose two forces acting perpendicularly on two equal arms of a bent lever to turn it opposite ways, these forces will balance, whatever be the angle which they make, since there is no reason why either should preponderate: but it would thus appear, that the force which would be balanced by Q in the figure to Axiom 7, would also be balanced by R, and therefore these two forces produce the same effect; which is what the axioms asserts.

33. The same reasoning might be applied to Axiom 9; for if two equal forces act at right angles at equal arms, in planes perpendicular to the axis of a rigid body, and tend to turn it opposite ways, they will balance each other, since all the conditions are the same for both.

34. Nearly the same might be said of Axiom 10;—if a string pass freely round a fixed body, equal forces acting at its two ends will balance each other; for if it pass with perfect freedom, its passing round the point cannot give an advantage to either force. Therefore the force which will be balanced by the string at its second extremity is exactly equal to the force which acts at its first extremity.*

35. The axioms which are perhaps least obvious are Axioms 4 and 5; for instance, the former;—that "the pressure upon the fulcrum is equal to the sum of the weights." Yet this becomes evident when we

* The same principle may be applied to prove Ax. 6.

consider it steadily. It will then be seen that we consider pressure or weight as something which must be supported, so that the whole support must be equal to the whole pressure. The two weights which act upon the lever must be somehow balanced and counteracted, and the length of the lever cannot at all remove or alter this necessity. Their pressure will be the same as if the two arms of the lever were shortened till the weights coincided at the fulcrum; but in this case, it is clear that the pressure on the fulcrum would be equal to the sum of the weights: therefore it will be so in every other case.

36. This principle, that in statical equilibrium, a force is necessarily supported by an equal force, is expressed in Axiom 1, with regard to forces acting at any point; and the two forces are then called action and re-action. The principle as stated in Axiom 1 may be considered as an expression of the conception of equality as applied to forces, or, if any one chooses, as a definition of equal forces. This principle is implied in the conception of any comparison of forces: for equilibrium and addition of forces are modes in which forces are compared, as superposition and addition of spaces are modes in which geometrical quantities are compared.

We may further observe, that this fundamental conception of action and re-action is equivalent to the conception of force and matter, which are ideas necessarily connected and correlative. Matter, as stated in page 26, is that which can resist the action of force. In Mechanics at least, we know matter only as the subject on which force acts.

37. But matter not only receives, it also transmits the action of force; and it is impossible to reason

respecting the mechanical results of such transmission, without laying down the fundamental principles by which it operates. And this accordingly is the purpose of Axioms 7, 8, 9, 10, 13. When the body is supposed to be perfectly rigid, it transmits force without any change or yielding. This rigidity of a body is contemplated under different aspects, in the Axioms just referred to. In Axiom 8, it is the rigidity of a rod pushed endways; in Axiom 7, the rigidity of a plane turned about a fixed point; in Axiom 9, the rigidity of a solid twisted about an axis. Axiom 10 defines the manner in which a flexible string transmits pressure, and in like manner Axiom 1 of the Hydrostatics, defines the manner in which a fluid transmits pressure. Any one who chooses may call Axioms 7, 8, 9 of the Statics, collectively, the Definition of a rigid body. The place of these principles in our reasoning will not be thereby altered; nor the necessity superseded, of their being accompanied by distinct mechanical conceptions.

38. Axioms 14, 15, 16, of the Statics, are all included in the general consideration that material bodies may be supposed to consist of material parts, and that the weight of the whole is equal to the weight of all the parts; but they are stated separately, because they are used separately, and because they are at least as evident in these more particular cases as they are in the more general form.

By considerations of this nature it appears, and I trust quite satisfactorily, that the axioms, as above stated, are evident in their nature, in virtue of the conceptions which we necessarily form, in order to reason upon mechanical subjects.

39. Some persons may be surprised to find the Axioms of Mechanics represented as so numerous: especially if they look for analogy to Geometry, where the necessary axioms are confessedly few, and according to some writers, none; and they may be led to think that many of the axioms here given must be superfluous, by observing that in most mechanical works the fundamental principles are stated as much fewer than these. But I believe that very few of those which I have stated are superfluous *in effect*. From the very circumstance that they *are* axioms, they are assented to when they are adduced in the reasoning, whether they have been before asserted or not; but to make our reasoning formally correct (which was one of my objects) every proposition which is assumed should be previously stated. And when we examine them, we see that the various modifications and combinations of the ideas of force, body, and equilibrium, along with the ideas of space of one, two, or three dimensions, readily branch out into as many heads as appear in this part of the present work.

40. Some persons may be disposed at first to say, that our knowledge of such elementary truths as are stated in the Axioms of Statics and Hydrostatics, is collected *from observation and experience*. But in refutation of this I remark, that we cannot experimentally verify these elementary truths, without assuming other principles which require proof as much as these do. If, for instance, Archimedes had wished to ascertain by trial whether two equal weights at the equal arms of a lever would balance each other, how could he know that the weights *were* equal, by any more simple criterion than that they *did* balance? But in fact, it is perfectly certain that of the thou-

sands of persons who from the time of Archimedes to the present day have studied Statics as a mathematical science, a very few have received or required any confirmation of his axioms from experiment; and those who have needed such help have undoubtedly been those in whom the apprehension of the real nature and force of the evidence of the subject was most obscure.

41. I by no means intend to assert that the axioms as stated in this Treatise are given in the only exact form; or that they may not be improved, simplified, and reduced in number. But I do not think it likely that this can be done to any great extent, consistently with the rigour of deductive proof. The Fourth Axiom of Statics is one which attempts have been made to supersede: for example, Lagrange* has endeavoured to deduce it from the preceding ones. But it will be found that his proof, if distinctly stated, involves some such axiom as this:—that "If two forces, acting at the extremities of a straight line, and a single force, acting at an intermediate point of the straight line, produce the same effect to turn a body about another line, the two forces produce at the intermediate point an effect equal to the single force." And though this axiom may be self-evident, it will hardly be considered as more simple than that which it replaces.

42. Thus, Statics, like Geometry, rests upon axioms which are neither derived directly from experience, nor capable of being superseded by definitions, nor by simpler principles. In this science, as in that previously considered, the evidence of these fundamental truths resides in those convictions, to which an attentive and steady consideration of the

* Mécanique Analytique. Introduction.

subject necessarily leads us. The axioms with regard to pressures, action and re-action, equilibrium and preponderance, rigid and flexible bodies, result necessarily from the conceptions which are involved in all exact reasoning on such matters. The axioms do not flow *from* the definitions, but they flow irresistibly *along with* the definitions, from the distinctness of our ideas upon the subjects thus brought into view. These axioms are not arbitrary assumptions, nor selected hypotheses; but truths which we must see to be necessarily and universally true, before we can reason on to anything else; and here, as in Geometry, the capacity of seeing that they are thus true, is required in the student, in order that he and the writer may be able to proceed together.

43. It was stated that the Axioms of Geometry are derived from the idea of space; in like manner the Axioms of Statics are derived from the *idea of statical force* or *pressure*, and the *idea of body* or *matter*, which, as we have said, is correlative with the idea of force. We must possess distinctly this idea of force acting upon body and body sustaining force;—of body resisting, and while it resists, transmitting the action of force;—of body, with this mechanical property, in the various forms of straight line, lever, plane, solid, flexible line, flexible surface, and fluid; and if we possess distinctly the ideas thus pointed out, the truth of the Axioms of Statics and Hydrostatics will be seen as self-evident, and we shall be in a condition to go on with the reasonings of the preceding Treatise, seeing both the cogency of the proof, and its necessary and independent character.

44. As the Axioms which are the basis of the Statics of Solids depend upon the idea of body, con-

sidered as transmitting force, so the axioms of Hydrostatics depend on the idea of a fluid, considered as a body which transmits pressure in all directions; or, as we may express it more briefly, upon the *idea of fluid pressure*. It is not enough to conceive a fluid as a body, the parts of which are perfectly moveable; for, as I have elsewhere observed*, "this definition cannot be a sufficient basis for the doctrines of the pressure of fluids; for how can we evolve, out of the mere notion of mobility, which includes no conception of force, the independent conception of pressure." But the conception of fluid as transmitting pressure, supplies us with the requisite axioms. The First Axiom of our Hydrostatics—that if a fluid be contained in a tube of which the two ends are similar and equal planes acted on by equal pressures, it will be kept in equilibrium—follows from the principle of sufficient reason, for there is no reason why either pressure should preponderate. If, for example, the curvature of the tube, or any such cause, affected the pressure at either end, this condition would be a limitation of the property of transmitting pressure in all directions, and would imply imperfect fluidity; whereas the fluidity is supposed to be *perfect*. And for the like reasons, we might assume as an *Axiom* the Third *Proposition* of the Hydrostatics, that fluids transmit pressure *equally* in all directions, from one part of their boundary to the other; for if the pressure transmitted were different according to the direction, this difference might be referred to some cohesion or viscosity of the fluid; and the fluidity might be made more perfect, by conceiving the difference removed. Therefore the proposition would be necessarily and evidently true of a perfect fluid.

* Thoughts on the Study of Mathematics.

45. But instead of laying down this axiom, I have taken the axiom that any part of a fluid which is in equilibrium, may be supposed to become rigid. This axiom leads immediately to the proposition, and it is, besides, of great use in all parts of Hydrostatics. If we had to reason concerning flexible bodies, we might conveniently and properly assume a corresponding axiom for them;—namely, that, of a flexible body which is in equilibrium, any part may be supposed to become rigid. And we might give a reason for this, by saying that rigidity implies forces which resist a tendency to change of form, when any such tendency occurs; but in a body which is in equilibrium, there is no tendency to change of form, and therefore the resisting forces vanish. It is of no consequence what forces *would* act *if there were* a stress to bend the body: since there *is not* any such stress, the rigidity is not called into play, and therefore it makes no difference whether we suppose it to exist or not.

46. The same kind of reasons may be given, in order to shew the admissibility of introducing, in the case of equilibrium of a fluid, rigidity, instead of that still greater susceptibility of change of figure which fluidity implies. Since the mass is perfectly fluid, its particles exert no constraint on each other's motions; but then, because they are in equilibrium, no constraint is needed to keep them in their places. They are as steadily kept there (so long as the same forces continue to act) as if they were held by the insurmountable forces which connect the parts of a perfectly rigid body. We may therefore suppose the inoperative forces of rigidity to be present or absent among the particles, without altering the other forces or their relations. And hence we see the truth of Axiom 2 of the Hydrostatics.

47. The above considerations (Art. 44) arising from the properties which we assume being perfect, may be applied in other cases; for instance, to shew that the force exerted by a *perfectly* smooth surface is perpendicular to the surface. (B. I. Ax. 13.) For if it were not, the force might be resolved into a force perpendicular to the surface, and a force acting along the surface; and the latter force might be referred to some friction or cohesion of the surface. Therefore we should not have supposed the surface perfectly smooth, without imagining this force to vanish: and thus the only force exerted by such a perfectly smooth surface would necessarily be a normal force.

48. The last axiom of Hydrostatics (Ax. 7) is in fact a substitute for an idea which we must exclude in Elementary Mathematics;—the idea of a *Limit*. The attempt to proceed far in Geometry without the use of this idea, gave rise to a series of well-known embarrassments among the ancients. The mode of evading the difficulty which I have adopted, by means of the axiom just referred to, appeared to me the best. The axiom is readily assented to, if it be considered that, since we may make the particles as small as we please, we may make as small as we please the error arising from the neglect of one particle. We may make it microscopic, and then throw away the microscope; and thus the error vanishes.

49. Some of the Axioms which are stated in Book III, on the Laws of Motion, give occasion to remarks similar to those already made. Thus Axiom 4, which asserts that if particles move in such a manner as always to preserve the same relative distances and positions, their motions will not be altered by supposing them rigidly connected, is evident by the

same considerations as the Axioms concerning flexible and fluid bodies, already noticed in Articles 45 and 46. For the forces of rigidity are forces which would prevent a change of the distances and relative positions of the particles if there were a tendency to any such change; and if there be no such tendency, it makes no difference whether the potential resistance to it be present or absent.

50. The 5th Axiom of Book III., which asserts that forces producing parallel and equal velocities at the same time, may be conceived to be added; and the 6th Axiom, which asserts that in systems in motion the action and re-action are equal and opposite, are applications of what is stated in the second sentence of this third Book;—that the Definitions and Axioms of Statics are adopted and assumed in the case of bodies in motion. In the third Book, as in the first, forces are conceived as capable of addition, and matter is conceived as that which can resist force, and transmit it unaltered.

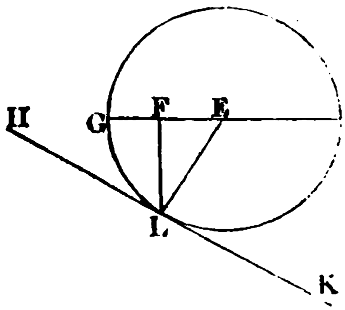
The 3d, 8th, and 9th Axioms of Book III., like the 7th of Book II., are introduced to avoid the reasoning which depends on Limits.

51. In the case of Mechanics, as in the case of Geometry, the distinctness of the idea is necessary to a full apprehension of the truth of the axioms; and in the case of mechanical notions it is far more common than in Geometry, that the axioms are imperfectly comprehended, in consequence of the want of distinctness and exactness in men's ideas. Indeed this indistinctness of mechanical notions has not only prevailed in many individuals at all periods, but we can point out whole centuries, in which it has been, so far as we can trace, universal. And the conse-

quence of this was, that the science of Statics, after being once established upon clear and sound principles, again fell into confusion, and was not understood as an exact science for two thousand years, from the time of Archimedes to that of Galileo and Stevinus.

52. In order to illustrate this indistinctness of mechanical ideas, I shall take from an ancient Greek writer an attempt to solve a mechanical problem; namely, the Problem of the Inclined Plane. The following is the mode in which Pappus professes* to answer this question:—"To find the force which will support a given weight A upon an inclined plane."

Let HK be the plane; let the weight A be formed into a sphere: let this sphere be placed in contact with the plane HK, touching it in the point L, and let E be its center. Let EG be a horizontal radius, and LF a vertical line which meets it. Take a weight B which is to A as EF to FG. Then if A and B be suspended at E and G to the lever EFG of which the center of motion is F, they will balance; being supported, as it were, by the fulcrum LF. And the sphere, which is equal to the weight A, may be supposed to be collected at its center. If therefore B act at G, the weight A will be supported.



It may be observed that in this attempt, the confusion of ideas is such, that the author assumes a

* Pappus, B. VIII. Prop. ix. I purposely omit the confusion produced by this author's mode of treating the question, in which he inquires the force which will *draw* a body *up* the inclined plane.

weight which acts at G, on the lever EFG, and which is therefore a vertical force, as identical with a force which acts at G, to support the body in the inclined plane, and which is parallel to the plane.

53. When this kind of confusion was remedied, and when men again acquired distinct notions of pressure, and of the transmission of pressure from one point to another, the science of Statics was formed by Stevinus, Galileo, and their successors*.

The fundamental ideas of Mechanics being thus acquired, and the requisite consequences of them stated in axioms, our reasonings proceed by the same rigorous line of demonstration, and under the same logical rules as the reasonings of Geometry; and we have a science of Statics which is, like Geometry, an exact deductive science.

SECT. II. *On the Logic of Induction.*

54. There are other portions of Mechanics which require to be considered in another manner; for in these there occur principles which are derived directly and professedly from experiment and observation. The derivation of principles by reasoning from facts is performed by a process which is termed *Induction*, which is very different from the process of Deduction already noticed, and of which we shall attempt to point out the character and method.

It has been usual to say of any general truths, established by the consideration and comparison of several facts, that they are obtained by *Induction*; but the distinctive character of this process has not been well pointed out, nor have any rules been laid

* See History of the Inductive Sciences, B. VI. chap. I. sect. 2, On the Revival of the Scientific Idea of Pressure.

down which may prescribe the form and ensure the validity of the process, as has been done for Deductive reasoning by common Logic. The *Logic of Induction* has not yet been constructed; a few remarks on this subject are all that can be offered here.

55. The Inductive Propositions, to which we shall here principally refer as examples of their class, are those elementary principles which occur in considering the motion of bodies, and of which some are called the Laws of Motion*. They are such as these;—a body not acted on by any force will move on for ever uniformly in a straight line;—gravity is a uniform force;—if a body in motion be acted upon by any force, the effect of the force will be compounded with the previous motion;—when a body communicates motion to another directly, the momentum lost by the first body is equal to the momentum gained by the second. And I remark, in the first place, that in collecting such propositions from facts, there occurs a step corresponding to the term "Induction," (ἐπαγωγὴ, *inductio*). Some notion is *superinduced* upon the observed facts. In each inductive process, there is some general idea introduced, which is given, not by the phenomena, but by the mind. The conclusion is not contained in the premises, but includes them by the introduction of a new generality. In order to obtain our inference, we travel beyond the cases we have before us; we consider them as exemplifications of, or deviations from, some ideal case in which the relations are complete and intelligible. We take a standard, and measure the facts by it; and this standard is created by us, not offered by Nature.

* Inductive Propositions in this work are, Book II. Propositions 25, 26, 32, 36, 37: Book III. Prop. 2, 3, 8, 13.

Thus we assert, that a body left to itself will move on with unaltered velocity, not because our senses ever disclosed to us a body doing this, but because (taking this as our ideal case) we find that all actual cases are intelligible and explicable by means of the notion of forces which cause change of motion, and which are exerted by surrounding bodies. In like manner, we see bodies striking each other, and thus moving, accelerating, retarding, and stopping each other; but in all this, we do not, by our senses, perceive that abstract quantity, momentum, which is always lost by one as it is gained by another. This momentum is a creation of the mind, brought in among the facts, in order to convert their apparent confusion into order, their seeming chance into certainty, their perplexing variety into simplicity. This the idea of momentum gained and lost does; and, in like manner, in any other case in which inductive truths are established, some idea is introduced, as the means of passing from the facts to the truth.

56. The process of mind of which we here speak can only be described by suggestion and comparison. One of the most common of such comparisons, especially since the time of Bacon, is that which speaks of induction as the *interpretation* of facts. Such an expression is appropriate; and it may easily be seen that it includes the circumstance which we are now noticing;—the superinduction of an idea upon the facts by the interpreting mind. For when we read a page, we have before our eyes only black and white, form and colour; but by an act of the mind, we transform these perceptions into thought and emotion. The letters are nothing of themselves; they contain no truth, if the mind does not contribute its share:

for instance, if we do not know the language in which the words are written. And if we are imperfectly acquainted with the language, we become very clearly aware how much a certain activity of the mind is requisite in order to convert the words into propositions, by the extreme effort which the business of interpretation requires. Induction, then, may be conveniently described as the interpretation of phenomena.

57. But I observe further, that in thus inferring truths from facts, it is not only necessary that the mind should contribute to the task its own idea, but, in order that the propositions thus obtained may have any exact import and scientific value, it is requisite that the idea be perfectly *distinct* and precise. If it be possible to obtain some vague apprehension of truths, while the ideas in which they are expressed remain indistinct and ill-defined, such knowledge cannot be available for the purposes we here contemplate. In order to construct a science, all our fundamental ideas must be distinct; and among them, those which Induction introduces.

58. This necessity for distinctness in the ideas which we employ in Induction, makes it proper to *define*, in a precise and exact manner, each idea when it is thus brought forwards. Thus, in establishing the propositions which we have stated as our examples in these cases, we have to define *force* in general; *uniform force*; *compounding* of motions; *momentum*. The construction of these definitions is an essential part of the process of Induction, no less than the assertion of the inductive truth itself.

59. But in order to justify and establish the inference which we make, the ideas which we introduce

must not only be distinct, but also *appropriate*. They must be exactly and closely applicable to the facts; so that when the idea is in our possession, and the facts under our notice, we perceive that the former includes and takes up the latter. The idea is only a more precise mode of apprehending the facts, and it is empty and unmeaning if it be anything else; but if it be thus applicable, the proposition which is asserted by means of it is true, precisely because the facts *are* facts. When we have defined force to be the cause of change of motion, we see that, as we remove external forces, we do, in actual experiments, remove all the change of motion; and therefore the proposition that there is in bodies no internal cause of change of motion, is true. When we have defined momentum to be the product of the velocity and quantity of matter, we see that in the actions of bodies, the effect increases as the momentum increases; and by measurement, we find that the effect may consistently be measured by the momentum. The ideas here employed are not only distinct in the mind, but applicable in the world: they are the elements, not only of relations of thought, but of laws of nature.

60. Thus an inductive inference requires an idea from within, facts from without, and a coincidence of the two. The idea must be distinct, otherwise we obtain no scientific truth; it must be appropriate, otherwise the facts cannot be steadily contemplated by means of it; and when they are so contemplated, the Inductive Proposition must be seen to be verified by the evidence of sense.

It appears from what has been said, that in establishing a proposition by Induction, the definition of the idea and the assertion of the truth, are not only both requisite, but they are correlative. Each of the

two steps contains the verification and justification of the other. The proposition derives its meaning from the definition; the definition derives its reality from the proposition. If they are separated, the definition is arbitrary or empty, the proposition is vague or verbal.

61. Hence we gather, that in the Inductive Sciences, our Definitions and our Elementary Inductive Truths ought to be introduced together. There is no value or meaning in definitions, except with reference to the truths which they are to express. Discussions about the definitions of any science, taken separately, cannot therefore be profitable, if the discussion do not refer, tacitly or expressly, to the fundamental truths of the science; and in all such discussions it should be stated what are taken as the fundamental truths. With such a reference to Elementary Inductive Truths clearly understood, the discussion of Definitions may be the best method of arriving at that clearness of thought, and that arrangement of facts, which Induction requires.

I will now note some of the differences which exist between Inductive and Deductive Reasoning, in the modes in which they are presented.

62. One leading difference in these two kinds of reasoning is, that in Deduction we infer particular from general truths; in Induction, on the contrary, we infer general from particular. Deductive proofs consists of many steps, in each of which we apply known general propositions in particular cases;—"all triangles have their angles equal to two right angles, therefore this triangle has; therefore, &c." In Induction, on the other hand, we have a single step in which we pass from many particular Propositions to

one general proposition; "This stone falls downwards; so do those others;—all stones fall downwards." And the former inference flows necessarily from the relation of general and particular; but the latter, as we have seen, derives its power of convincing from the introduction of a new idea, which is distinct and appropriate, and which supplies that generality which the particulars cannot themselves offer.

63. I observe also that this difference of process in inductive and deductive proofs, may be most properly marked by a difference in the form in which they are stated. In Deduction, the *Definition* stands at the beginning of the proposition; in Induction, it may most suitably stand at or near the end. Thus the definition of a uniform force is introduced in the course of the proposition that gravity is a uniform force. And this arrangement represents truly the real order of proof; for, historically speaking, it was taken for granted that gravity was a uniform force; but the question remained, what was the right definition of a uniform force. And in the establishment of other inductive principles, in like manner, definitions cannot be laid down for any useful purpose, till we know the propositions in which they are to be used. They may therefore properly come each at the conclusion of its corresponding proposition.

64. The ideas and definitions which are thus led to by our inductive process, may bring with them Axioms. Such Axioms may be self-evident as soon as the inductive idea has been distinctly apprehended, in the same manner as was explained respecting the fundamental ideas of Geometry and Statics. And thus *Axioms*, as well as Definitions, may come at the end of our Inductive Propositions; and they thus

assume their proper place at the beginning of the deductive propositions which follow them, and are proved from them. Thus, in Book III., Axioms 8 and 9, come after the definition of Accelerating Force, and stand between Props. 14 and 15.

65. Another peculiarity in inductive reasoning may be noticed. In a deductive demonstration, the reference is always to what has been already proved; in establishing an Inductive Principle, it is most convenient that the reference should be to subsequent propositions. For the proof of the Inductive Principle consists in this;—that the principle being adopted, consequences follow which agree with fact; but the demonstration of these consequences may require many steps, and several special propositions. Thus the Inductive Principle, that gravity is a uniform force, is established by shewing that the law of descent, which falling bodies follow in fact, is explained by means of this principle; namely, the law that the space is as the square of the time from the beginning of the motion. But the proof of such a property, from the definition of a uniform force, requires many steps, as may be seen in the preceding Treatise, Book III. Prop. 5: and this proof must be referred to, along with several others, in order to establish the truth, that gravity is a uniform force.

66. It may be suggested, that, this being the case, the propositions might be transposed, so that the inductive proof might come after those propositions to which it refers. But if this were done, all the propositions which depend upon the laws of motion must be proved hypothetically only. For instance, we must say, "If, in the communication of motion, the momentum lost and gained be equal, the velocity acquired

by a body falling down an inclined plane, will be equal to that acquired by falling down the height." This would be inconvenient, and even if it were done, that completeness in the line of demonstration which is the object of the change, could not be obtained; for the transition from the particular cases to the general truth, which must occur in the Inductive Proposition, could not be in any way justified according to rules of Deductive Logic.

I have, therefore, in the preceding pages, placed the Inductive Principle first in each line of reasoning; and have ranged after it the Deductions from it, which justify and establish it as their first office, but which are more important as its consequences and applications, after it is supposed to be established.

67. I have used one common *formula* in presenting the proof of each of the Inductive Principles which I have introduced;—namely, after stating or exemplifying the facts which the induction includes, I have added "These results can be clearly explained and rigorously deduced by introducing the *Idea* or the *Definition*," which belongs to each case, "and the *Principle*," which expresses the inductive truth. I do not mean to assert that this formula is the only right one, or even the best; but it appears to me to bring under notice the main circumstances which render an induction systematic and valid.

68. It may be observed, however, that this formula does not express the full cogency of the proof. It declares only that the results *can* be clearly explained and rigorously deduced by the employment of a certain definition and a certain proposition. But in order to make the conclusion demonstrative, we ought to be able to declare that the results can be clearly ex-

plained and rigorously deduced *only* by the definition and proposition which we adopt. And, in reality, the mathematician's conviction of the truth of the Laws of Motion does depend upon his seeing that they (or laws equivalent to them) afford the *only* means of clearly expressing and deducing the actual facts. But this conviction, that no other law than those proposed can account for the known facts, finds its place in the mind gradually, as the contemplation of the consequences of the law and the various relations of the facts becomes steady and familiar. I have therefore not thought it proper to require such a conviction along with the first assent to the inductive truths which I have here stated.

69. The propositions established by Induction are termed *Principles*, because they are the starting points of trains of deductive reasoning. In the system of deduction, they occupy the same place as axioms; and accordingly they are termed so by Newton—"Axiomata sive leges motus." Stewart objects strongly to this expression*: and it would be difficult to justify it; although to draw the line between axioms and inductive principles may be a harder task than at first appears.

70. But from the consideration that our Inductive Propositions are the principles or beginnings of our deductive reasoning, and so far at least stand in the place of axioms, we may gather *this* lesson,—that they are not to be multiplied without necessity. For instance, if in a treatise on Hydrostatics, we should state as two separate propositions, that "air has weight;" and that "the mercury in the barometer is sustained by the weight of the air;" and should prove both the one

* Elem. Phil. Human Mind. Vol. II. p. 44.

and the other by reference to experiment; we should offend against the maxims of Logic. These propositions are connected; the latter may be demonstrated deductively from the former; the former may be inferred inductively from the facts which prove the latter. One of these two courses ought to be adopted; we ought not to have two ends of our reasoning upwards, or two beginnings of our reasoning downwards.

71. I shall not now extend these Remarks further. They may appear to many barren and unprofitable speculations; but those who are familiar with such subjects, will perhaps find in them something which, if well founded, is not without some novelty for the English reader. Such will, I think, be the case, if I have satisfied him,—that mathematical truth depends on axioms as well as definitions,—that the evidence of geometrical axioms is to be found only in the distinct possession of the idea of space,—that other branches of mathematics also depend on axioms,—and that the evidence of these axioms is to be sought in some appropriate idea;—that the evidence of the axioms of statics, for instance, resides in the ideas of force and matter;—that in the process of induction the mind must supply an idea in addition to the facts apprehended by the senses;—that in each such process we must introduce one or more definitions, as well as a proposition;—that the definition and the proposition are correlative, neither being useful or valid without the other;—and that the formula of inductive reasoning must be in many respects the reverse of the common logical formulæ of deduction.

ERRATA.

Page 20, line 1; *for* 8, 12, 16 *read* 8, 12, 18.

—— 39, second diagram; below N insert Q.

— 62, line 15 *read*, we exchange the line HK for the line GF.

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PREFACE TO THE FOURTH EDITION.

"The Mechanical Euclid" is a title which, perhaps requires some apology, since the word "Euclid" is here used to signify, not a person, but a system of elementary propositions, connected and demonstrated with a rigour like that of the Elements of Geometry. The work was undertaken from a conviction that, if it could be properly executed, the sciences of Mechanics and Hydrostatics might be employed, as well as Geometry, in that discipline of the mind which is an essential part of a sound education, and of which rigorous mathematical reasoning is so important and valuable an instrument. And since the University of Cambridge has recently declared itself of this opinion, by appointing the elementary portions of Mechanics and Hydrostatics as a necessary part of the ordinary examinations for degrees, the work has been carefully adapted to the scheme thus laid down by authority.

In an elementary science thus intended to be employed as a discipline of the intellect, it is desirable that the matter to be studied should be reduced to certain distinct and fixed Propositions, as is done in Geometry. I have therefore, in this Edition, adopted the list of Propositions in Mechanics and Hydrostatics, required by the University in the examination above

mentioned; and have preserved the numbers of that list without change; marking the few additional Propositions which I have introduced with the letters of the alphabet, as is done by Simson in his Euclid. When the existing scheme of University Examination has been continued a few years longer, it may be hoped that this list of Elementary Propositions in Mechanics and Hydrostatics will become *classical*, as the Propositions of Euclid's Elements are: so that "the eighth Proposition of Mechanics," or "the sixth of Hydrostatics," may be expressions as familiarly understood as "the forty-seventh of Euclid's First Book," or "the fourth of his Second."

So far as I have learnt, the Examination in the Elements of Mechanics and Hydrostatics thus appointed by the University, has, in its operation, shewn a highly satisfactory prospect of the beneficial effects which it is likely to produce when its course shall have been well determined by practice. Perhaps I may be allowed to make here one or two remarks bearing upon this subject.

One ground on which some persons may perhaps for a moment doubt the efficacy of this examination as an intellectual discipline, is this:—that the list of Propositions being thus limited and known beforehand, there seems to be nothing to prevent the student from learning the demonstrations by rote, and delivering them to the examiner without understanding them. And to this I reply, that the same argument might be urged, with at least equal force, against the value

of Euclid's Geometry as a part of our examinations; and yet I believe every one practically acquainted with University and College examinations and their effects, will agree with me that Euclid's Geometry is the most effective and the most valuable portion of our mathematical education. If the examinations in Mechanics and Hydrostatics be assimilated as much as possible to the examinations in Euclid, they will have the same kind of effect, as a discipline of strict reasoning; and the study of these additional sciences will bring with it additional advantages, arising from the more extensive and varied nature of the subjects thus presented to the student's mind.

In introducing these additional sciences into the study for the usual degree, the portion which Algebra occupied in the examinations was rather diminished than increased. So far as this change was requisite to facilitate the introduction of the new portions of the examination, it will not, I think, be deemed an evil by any one who wishes the studies of the University to be so selected and arranged as to be an intellectual discipline. For the knowledge of Algebra which is generally acquired by those who study that subject merely with a view to the ordinary degree, must be so scanty as to be of small value for the purpose just mentioned; especially when we take into account the very imperfect acquaintance with Arithmetic which students in general, according to the present practice of many places of previous education, bring to the University. Even in the hands of those who are able

to use it with facility and certainty, as a language and an instrument, the great charm of Algebra is that it expresses reasonings, and obtains the result of them, without the exercise of the reason: and when students are required to follow in a general form relations and combinations of numbers which they cannot deal with in particular cases, their apprehension of the meaning and grounds of the processes must be so obscure, as to prevent the mind receiving any portion of the salutary effect which a complete mastery of the science might produce.

There are indeed a few simple algebraical terms and operations which occur so familiarly in mathematical reasonings, that the student cannot conveniently remain ignorant of them; and accordingly, the University has directed that the examination above mentioned shall include questions of this kind. These parts of algebra, extracted from Dr. Wood's Algebra by permission of the author, are given in the Introduction to the present work, along with a few other portions of Pure Mathematics, to which it is convenient to be able to refer in a succeeding part of the book.

Some of the enunciations of theorems contained in the Schedule sanctioned by the University, (in consequence, I conceive, of the wish felt by the framers of the plan that the document should be as brief as possible,) contain Propositions each of which may conveniently be separated into two or more; for instance, Prop. VIII, and XVI, of the Mechanics; and Prop. I,

II, V, VI, X, of the Hydrostatics. Perhaps it might be convenient, when these Propositions are required in an examination, to state which *Case* is intended.

In order that the present little work may serve as guide to the student in preparing for the examination to which I have referred, I have inserted in an Appendix the Grace of Feb. 22, 1837, (by which this part of the examination was founded,) as modified by the Grace of May 11, 1842.

TRIN. COLL.
March 13, 1843

PREFACE TO THE FIFTH EDITION.

In this edition I have restored to their places the *Third Book of Mechanics*, which contains the Laws of Motion, and the *Remarks on Mathematical Reasoning*, and *on the Logic of Induction*; both which portions were, with a view to brevity, omitted in the fourth edition; but are, it appears, desired by many readers. I have also inserted the Questions on Mechanics and Hydrostatics proposed in the Examinations for the present year; and the modifications of the Regulations respecting the examinations which were introduced by the Grace of March 20, 1846.

It has been noticed to me that the demonstration of Prop. VIII. B. 1, may be somewhat simplified in this manner.

After the words "and therefore DAq is a straight line," go on thus:

And therefore DA is parallel to Cp. Also since CAr is a straight line, CA is parallel to Dp. Hence DC is a parallelogram, and therefore $CA = Dp$. But since pr also is a parallelogram, $Dp = Ar$; therefore $CA = Ar$.

TRIN. COLL.
March 23, 1849.

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INTRODUCTION.

ELEMENTARY PURE MATHEMATICS.

ALGEBRA.

*(1). *To define and explain Algebraical Signs.*

Art. 1. The method of representing the relation of abstract quantities by letters and characters, which are made the signs of such quantities and their relations, is called Algebra.

Known or determined quantities are usually represented by the first letters of the alphabet a, b, c, d, &c. and unknown or undetermined quantities by the last y, x, w, &c.

The following signs are made use of to express the relations which the quantities bear to each other.

2. + *Plus*, signifies that the quantity to which it is prefixed must be added. Thus $a + b$ signifies that the quantity represented by b is to be added to the quantity represented by a; if a represent 5, and b, 7, then $a + b$ represents 12.

If no sign be placed before a quantity, the sign + is understood. Thus a signifies $+ a$. Such quantities are called positive quantities.

3. − *Minus*, signifies that the quantity to which it is prefixed must be subtracted. Thus $a - b$ signifies that b must be taken from a; if a be 7, and b, 5, $a - b$ expresses 7 diminished by 5, or 2.

Quantities to which the sign − is prefixed are called negative quantities.

4. $\times$ *Into*, signifies that the quantities between which it stands are to be multiplied together. Thus $a \times b$ signifies that the quantity represented by a is to be multiplied by the quantity represented by b*.

This sign is frequently omitted; thus $a\,b\,c$ signifies $a \times b \times c$, or a full point is used instead of it; thus $1 \times 2 \times 3$, and $1 . 2 . 3$ signify the same thing.

5. If in multiplication the same quantity be repeated any number of times, the product is usually expressed by placing above the quantity the number which represents how often it is repeated; thus a, $a \times a$, $a \times a \times a$, $a \times a \times a \times a$, and a^1, a^2, a^3, a^4, have respectively the same signification. These quantities are called *powers*; thus a^1, is called the *first power* of a; a^2, the *second power*, or *square* of a; a^3, the *third power*, or *cube* of a; a^4, the *fourth power*, or *biquadrate* of a. The succeeding powers have no names in common use except those which are expressed by means of number; thus a^7 is the *seventh power* of a, or *a to the seventh power*; and a^n is a to the n^{th} power.

The numbers 1, 2, 3, &c. are called the *indices* of a; or *exponents* of the powers of a.

6. $\div$ *Divided by*, signifies that the former of the quantities between which it is placed is to be divided by the latter. Thus $a \div b$ signifies that the quantity a is to be divided by b.

The division of one quantity by another is frequently represented by placing the dividend over the divisor with a line between them, in which case the expression is called *a fraction*. Thus, $\frac{a}{b}$ signifies a

* By quantities, we understand such magnitudes as can be represented by numbers; we may therefore without impropriety speak of the multiplication, division, &c. of quantities by each other.

divided by b; and a is the *numerator*, and b the *denominator* of the fraction; also $\frac{a+b+c}{e+f+g}$ signifies that a, b, and c added together, are to be divided by e, f, and g added together.

7. A quantity in the denominator of a fraction is also expressed by placing it in the numerator, and prefixing the negative sign to its index; thus a^{-1}, a^{-2}, a^{-3}, a^{-n} signify $\frac{1}{a^1}$, $\frac{1}{a^2}$, $\frac{1}{a^3}$, $\frac{1}{a^n}$ respectively; these are called the *negative powers* of a.

8. The *reciprocal* of a fraction is the fraction inverted. Thus $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$; and $\frac{1}{a}$ is the reciprocal of a.

9. A line drawn over several quantities signifies that they are to be taken collectively, and it is called a *vinculum*. Thus $\overline{a-b+c} \times \overline{d-e}$ signifies that the quantity represented by $a-b+c$ is to be multiplied by the quantity represented by $d-e$. Let a stand for 6; b, 5; c, 4; d, 3; and e, 1; then $a-b+c$ is $6-5+4$, or 5; and $d-e$ is $3-1$, or 2; therefore $\overline{a-b+c} \times \overline{d-e}$ is 5×2, or 10. $\overline{ab-cd} \times \overline{ab-cd}$ or $\overline{ab-cd}|^2$ signifies that the quantity represented by $ab-cd$ is to be multiplied by itself.

Instead of a line, brackets are sometimes used, as $(ab-cd)^2$, $\{a-b+c\} \cdot \{d-e\}$.

10. $=$ *Equal to*, signifies that the quantities between which it is placed are equal to each other, thus $ax-by=cd+ad$, signifies that the quantity $ax-by$ is equal to the quantity $cd+ad$.

11. The *square root* of any proposed quantity is that quantity whose square, or second power, gives the proposed quantity. The *cube root*, is that quantity whose cube gives the proposed quantity, &c.

The signs $\sqrt{\ }$, or $\sqrt[2]{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, &c. are used to express the square, cube, biquadrate, &c. roots of the quantities before which they are placed.

$$\sqrt{a^2} = a, \ \sqrt[3]{a^3} = a, \ \sqrt[4]{a^4} = a, \ \&c.$$

These roots are all represented by the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. placed a little above the quantities, to the right. Thus $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, $a^{\frac{1}{n}}$, represent the square, cube, fourth and n^{th} root of a, respectively; $a^{\frac{5}{2}}$, $a^{\frac{7}{3}}$, $a^{\frac{3}{5}}$, represent the square root of the fifth power, the cube root of the seventh power, the fifth root of the cube of a.

12. If these roots cannot be exactly determined, the quantities are called *irrational* or *surds*.

13. Points are made use of to denote *proportion*, thus $a : b :: c : d$, signifies that a bears the same proportion to b that c bears to d.

14. The number prefixed to any quantity, and which shews how often it is to be taken, is called its *coefficient*. Thus, in the quantities $7\,ax$, $6\,by$, and $3\,dz$, 7, 6, and 3 are called the coefficients of ax, by, and dz respectively.

When no number is prefixed, the quantity is to be taken once, or the coefficient 1 is understood.

These numbers are sometimes represented by letters, which are called *coefficients*.

15. Similar, or *like* algebraical quantities are such as differ only in their coefficients; $4\,a$, $6\,ab$, $9\,a^2$,

$3a^2bc$, are respectively similar to $15a$, $3ab$, $12a^2$, $15a^2bc$, &c.

Unlike quantities are different combinations of letters; thus, ab, a^2b, ab^2, abc, &c. are unlike.

16. A quantity is said to be a *multiple* of another, when it contains it a certain number of times exactly: thus $16a$ is a multiple of $4a$, as it contains it exactly four times.

17. A qnantity is called a *measure* of another, when the former is contained in the latter a certain number of times exactly; thus, $4a$ is a measure of $16a$.

18. When two numbers have no *common measure* but unity, they are said to be *prime* to each other.

19. A *simple* algebraical quantity is one which consists of a single term, as a^2bc.

20. A *binomial* is a quantity consisting of two terms, as $a + b$, or $2a - 3bx$. A *trinomial* is a quantity consisting of three terms, as $2a + bd + 3c$.

21. The following examples will serve to illustrate the method of representing quantities algebraically:—

Let $a = 8$, $b = 7$, $c = 6$, $d = 5$ and $e = 1$; then

$$3a - 2b + 4c - e = 24 - 14 + 24 - 1 = 33.$$

$$ab + ce - bd = 56 + 6 - 35 = 27.$$

$$\frac{a+b}{c-e} + \frac{3b-2c}{a-d} = \frac{8+7}{6-1} + \frac{21-12}{8-5}$$

$$= \frac{15}{5} + \frac{9}{3} = 6.$$

$$\times \overline{a-c} - 3cc^2 + d^3 = 25 \times 2 - 18 + 125$$

$$= 50 - 18 + 125 = 157.$$

(2). *To add and subtract simple Algebraical Quantities.*

22. The addition of algebraical quantities is performed by connecting those that are *unlike* with their proper signs, and collecting those that are *similar* into one sum.

Examples:

Add	Add
$4x$	$5ax$
$3x$	$-ax$
$7a$	by
$-2a$	$-cy$
Sum $7x+5a$	Sum $4ax+by-cy$

$a+2bx-y^2$	$a+3b$
$b-bx+3y^2$	$a+n-4b$
Sum $a+b+bx+2y^2$	Sum $2a+n-1b$

23. Subtraction, or the taking away of one quantity from another, is performed by changing the sign of the quantity to be subtracted, and then adding it to the other by the rules laid down in Art. 22.

From	$7x$	From	$7x+3a$
Subtract	x	Subtract	$5a-x$
Diff.	$7x-x$ or $6x$	Diff.	$7x+x+5a-3a$ or $8x+2a$

From	$4x^2+5ax-y^2$
Subtract	$3x^2-3ax+y^2$
Diff.	$x^2+8ax-2y^2$

*(3). *To multiply simple Algebraical Quantities.*

24. The multiplication of simple algebraical quantities must be represented according to the notation pointed out in Art. 4 and 5. Thus, $a \times b$, or ab, represents the product of a multiplied by b; abc, the product of the three quantities a, b, and c:

It is also indifferent in what order they are placed, $a \times b$ and $b \times a$ being equal.

25. If the quantities to be multiplied have coefficients, these must be multiplied together as in common arithmetic; the literal product being determined by the preceding rules.

Thus, $3a \times 5b = 15ab$; because

$$3 \times a \times 5 \times b = 3 \times 5 \times a \times b = 15ab.$$

26. The powers of the same quantity are multiplied together by adding the indices: thus, $a^2 \times a^3 = a^5$; for $aa \times aaa = aaaaa$. In the same manner,

$$a^m \times a^n = a^{m+n}; \text{ and } 3a^2x^3 \times 5axy^2$$
$$= 15a^3x^4y^2.$$

27. If the *multiplier* or *multiplicand* consist of several terms, each term of the latter must be multiplied by every term of the former, and the sum of all the products taken, for the whole product of the two quantities.

*(4). *To divide simple Algebraical Quantities.*

28. To divide one quantity by another, is to determine how often the latter is contained in the former, or what quantity multiplied by the latter will produce the former.

Thus, to divide ab by a is to determine how often a must be taken to make up ab; that is, what quantity multiplied by a will give ab; which we know

is b. From this consideration are derived all the rules for the division of algebraical quantities.

If only a part of the product which forms the *divisor* be contained in the *dividend*, the division must be represented according to the direction in Art. 6, and the quantities contained both in the divisor and dividend expunged.

Thus $15a^2b^2c$ divided by $3a^2bx$ is $\frac{15a^2b^2c}{3a^2bx}$, which is equal to $\frac{5bc}{x}$, expunging from the dividend and from the divisor the quantities 3, a^2, and b.

*(5). *To reduce Fractions to others of equal value which have a common denominator.*

29. Fractions are changed to others of equal value with a common denominator, by multiplying each numerator by every denominator except its own, for the new numerator; and all the denominators together for the common denominator.

Let $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$; be the proposed fractions; then $\frac{adf}{bdf}$, $\frac{cbf}{bdf}$, $\frac{edb}{bdf}$, are fractions of the same value with the former, have the common denominator bdf. For $\frac{adf}{bdf}=\frac{a}{b}$; $\frac{cbf}{bdf}=\frac{c}{d}$; and $\frac{edb}{bdf}=\frac{e}{f}$ (Art, 28); the numerator and denominator of each fraction having been multiplied by the same quantity viz.—the product of the denominators of all the other fractions.

30. When the denominators of the proposed fractions are not prime to each other, find their greatest common measure; multiply both the numerator and

denominator of each fraction by the denominators of all the rest, divided respectively by their greatest common measure; and the fractions will be reduced to a common denominator in lower terms* than they would have been by proceeding according to the former rule.

Thus $\frac{a}{mx}$, $\frac{b}{my}$, $\frac{c}{mz}$, reduced to a common denominator are $\frac{ayz}{mxyz}$ $\frac{bxz}{mxyz}$ $\frac{cxy}{mxyz}$

*(6). *To add together simple Algebraical Fractions.*

31. If the fractions to be added have a common denominator their sum is found by adding the numerators together and retaining the common denominator. Thus,

$$\frac{2a}{5}+\frac{a}{5}\quad 3a$$

$$\frac{a+2x}{3}+\frac{a-x}{3}=\frac{2a}{3}$$

$$\frac{7x+y}{a}+\frac{2y-5x}{a}\quad 2x-4y$$

32. If the fractions have not a common denominator, they must be transformed to others of the same value which have a common denominator, (by Art. 29), and then the addition may take place as before. Thus,

$$\frac{a}{3}+\frac{a}{5}=\frac{5a}{15}+\frac{3a}{15}\quad\frac{8a}{15}$$

$$\frac{a}{b}+\frac{a}{x}=\frac{ax}{bx}+\frac{ab}{bx}\quad\frac{ax+ab}{bx}$$

To obtain them in the *lowest* terms, each must be reduced to another of equal value, with the denominator which is the least common multiple of all the denominators.

$$\frac{a}{b} + 1 = \frac{a}{b} + \frac{b}{b} = \frac{a+b}{b};$$

$$2 - \frac{a}{3x} = \frac{6x - a}{3x}.$$

*(7). *To multiply simple Algebraical Fractions.*

33. To multiply a fraction by any quantity, multiply the numerator by that quantity and retain the denominator.

Thus $\frac{a}{b} \times c = \frac{ac}{b}$. For if the quantity to be divided be c times as great as before, and the divisor the same, the quotient must be c times as great.

34. The product of two fractions is found by multiplying the numerators together for a new numerator, and the denominators for a new denominator.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be the two fractions: then $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

For if $\frac{a}{b} = x$, and $\frac{c}{d} = y$, by multiplying the equal quantities $\frac{a}{b}$ and x by b, $a = bx$ (Art. 28), in the same manner $c = dy$; therefore, by the same axiom, $ac = bdxy$; dividing these equal quantities, ac and dxy by bd, we have $\frac{ac}{bd} = xy = \frac{a}{b} \times \frac{c}{d}$.

*(8). *To divide simple Algebraical Fractions.*

35. To divide a fraction by any quantity, multiply the denominator by that quantity, and retain the numerator.

The fraction $\frac{a}{b}$ divided by c, is $\frac{a}{bc}$. Because $\frac{a}{b} = \frac{ac}{bc}$, and a c^{th} part of this is $\frac{a}{bc}$; the quantity to be divided being a c^{th} part of what it was before, and the divisor the same.

36. To divide a quantity by any fraction, multiply the quantity by the reciprocal of the fraction. (Art. 8).

If we divide c by $\frac{a}{b}$ we obtain $\frac{bc}{a}$. For if

$$c \div \frac{a}{b} = x, \quad c = x \times \frac{a}{b}, \text{ or } c = \frac{ax}{b}, \text{ and } x = \frac{bc}{a}.$$

*(9). *Algebraical definition of Proportion.*

37. Four quantities are said to be proportionals, when the first is the same multiple, part, or parts of the second, that the third is of the fourth.

Thus the four quantities 8, 12, 6, 9, are proportionals; for 8 is $\frac{2}{3}$ of 12, and 6 is $\frac{2}{3}$ of 9.

In this case $\frac{8}{12} = \frac{6}{9}$; and generally a, b, c, d are proportionals if $\frac{a}{b} = \frac{c}{d}$. This is usually expressed by saying a is to b, as c to d; and thus represented, $a : b :: c : d$.

The terms a and d are called the *extremes*, and b and c the *means*.

The fraction $\frac{a}{b}$ is called the *ratio* of a to b.

*(10). *Algebraical consequences of Proportion.*

38. When $\frac{a}{b} = \frac{c}{d}$, if a be equal to b, c is equal to d, and if a be less than b, c is less than d, and if a be greater than b, c is greater than d.

39. When four quantities are proportionals, the product of the extremes is equal to the product of the means.

Let a, b, c, d be the four quantities; then, since they are proportionals, $\frac{a}{b} = \frac{c}{d}$; and by multiplying both sides by bd, $ad = bc$.

Any three terms in a proportion $a : b :: c : d$ being given, the fourth may be determined from the equation $ad = bc$.

40. If the first be to the second as the second to the third, the product of the extremes is equal to the square of the mean.

For (Art. 39) if $a : x :: x : b$, $ab = x^2$.

41. If the product of two quantities be equal to the product of two others, the four are proportionals, making the terms of one product the means, and the terms of the other the extremes.

Let $xy = ab$, then dividing by ay, $\frac{x}{a} = \frac{b}{y}$,

or, $x : a :: b : y$.

42. If $a : b :: c : d$, and $c : d :: e : f$, then will $a : b :: e : f$.

Because $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$, therefore $\frac{a}{b} = \frac{e}{f}$; or

$a : b :: e : f$.

43. If four quantities be proportionals, they are also proportionals when taken *inversely*.

If $a : b :: c : d$, then $b : a :: d : c$. For $\frac{a}{b} = \frac{c}{d}$, and dividing unity by each of these equal

quantities, or taking their reciprocals, $\frac{b}{a} = \frac{d}{c}$; (Art. 36)

that is, $b : a :: d : c$.

44. If four quantities be proportionals, they are proportionals when taken *alternately*.

If $a : b :: c : d$, then $a : c :: b : d$.

Because the quantities are proportionals, $\frac{a}{b} = \frac{c}{d}$;

and multiplying by $\frac{b}{c}$, $\frac{a}{c} = \frac{b}{d}$, or $a : c :: b : d$.

45. Unless the four quantities are of the same kind, the alternation cannot take place; because this operation supposes the first to be some multiple, part, or parts, of the third.

One line may have to another line the same ratio that one weight has to another weight, but a line has no relation in respect of magnitude to a weight. In cases of this kind, if the four quantities be represented by numbers or other quantities which are similar, the alternation may take place, and the conclusions drawn from it will be just.

46. If $a : b :: c : d$, then *componendo*,

$$a + b : b :: c + d : d.$$

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} + 1 = \frac{c}{d} + 1$;

therefore $\frac{a+b}{b} = \frac{c+d}{d}$;

therefore $a + b : b :: c + d : d$.

47. Also *dividendo*, $a - b : b :: c - d : d$.

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} - 1 = \frac{c}{d} - 1$;

$$\text{therefore } \frac{a-b}{b} = \frac{c-d}{d},$$

$$\text{therefore } a - b : b :: c - d : d.$$

48. Also *convertendo*, $a : a - b :: c : c - d$.

$$\text{For } \frac{a}{b} = \frac{c}{d}; \text{ therefore } \frac{b}{a} = \frac{d}{c};$$

$$\text{therefore } 1 - \frac{b}{a} = 1 - \frac{d}{c};$$

$$\text{therefore } \frac{a-b}{a} = \frac{c-d}{c}; \text{ therefore } a - b : a :: b - d : c;$$

and by Art. 43, $a : a - b :: c : c - d$.

49. If we have any number of sets of proportionals, and if the corresponding terms be multiplied together, the products are proportionals.

If $a : b :: c : d$, and $p : q :: r : s$,

and $u : v :: x : y$,

then $apu : bqv :: crx : dsy$.

$$\text{For } \frac{a}{b} = \frac{c}{d}, \text{ and } \frac{p}{q} = \frac{r}{s}, \text{ and } \frac{u}{v} = \frac{x}{y};$$

$$\text{and multiplying together equals } \frac{apu}{bqv} = \frac{crx}{dsy};$$

$$\text{therefore } apu : bqv :: crx : dsy.$$

50. If the same quantities occur in the antecedents of one set of proportionals and the consequents of another set, the resulting proportionals will be reduced.

If $a : b :: c : d$, and $b : e :: d : f$,

then $a : e :: c : f$.

$$\text{For } \frac{a}{b} = \frac{c}{d} \text{ and } \frac{b}{e} = \frac{d}{f}; \text{ therefore } \frac{ab}{be} = \frac{cd}{df},$$

and $\frac{a}{e} = \frac{c}{f}$; wherefore $a : e :: c : f$.

If $a : b :: x : y$, and $b : c :: z : x$,

and $c : d :: z : t$,

then $a : d :: z^2 : ty$.

For $\frac{a}{b} = \frac{x}{y}$, and $\frac{b}{c} = \frac{z}{x}$, and $\frac{c}{d} = \frac{z}{t}$;

therefore $\frac{abc}{bcd} = \frac{xzz}{yxt}$;

and expunging common factors in the numerators and denominators,

$$\frac{a}{d} = \frac{z^2}{yt}.$$

*(11). *Of Variation.*

51. Quantities of the same kind assume different values under constant conditions, and when these different values are compared, the quantities are spoken of as *variable*, and the proportion of the different values may be expressed by two terms of a proportion instead of four.

Thus if a man travel with a constant velocity (for example 4 miles an hour,) the space travelled over in any one time is to the space travelled over in any other time as the first time is to the second time; and this may be expressed by saying that the space *varies as* the time, or *is as* the time.

52. One quantity is said to *vary directly* as another when the two quantities depend wholly upon each other, in such a manner that if the one be changed the other is changed in the same proportion.

If the altitude of a triangle be invariable, the area varies as the base. For if the base be increased or diminished in any proportion, the area is increased or diminished in the same proportion. (Euc. VI. 1.)

53. One quantity is said to vary *inversely* as another, when the former cannot be changed in any manner, but the reciprocal of the latter is changed in the same manner.

If the area of a triangle be given the base varies as the perpendicular altitude.

If A, a represent the altitudes, B, b the bases of two triangles, since a triangle is half the rectangle on the same base and of the same altitude, and the triangles are equal, $\frac{1}{2}AB = \frac{1}{2}ab$. (See Geometry.)

Therefore

$$A : a :: b : B, \text{ or } A : a :: \frac{1}{B} : \frac{1}{b}.$$

54. One quantity is said to *vary as others jointly*, if, when the former is changed in any manner, the product of the others is changed in the same proportion.

The area of a triangle varies as its altitude and base jointly.

Let A, B, a, b be the altitudes and bases of two triangles as before, and S, s the areas; then

$$S = \tfrac{1}{2}AB,\ s = \tfrac{1}{2}ab \text{ and } S : s :: AB : ab.$$

55. In the same manner $A : a :: \frac{S}{B} : \frac{s}{b}$; and A varies as S directly and B inversely.

56. The symbol $\propto$ is often used for variation. Thus the above variations may be expressed

$$A \propto \frac{1}{B},\ S \propto AB,\ A \propto \frac{S}{B}.$$

57. When the increase or decrease of one quantity depends upon the increase or decrease of two others, and it appears that if either of these latter be constant, the first varies as the other, when they both vary, the first varies as their product.

Thus, if V be the velocity of a body moving uniformly, T the time of motion, and S the space described; if T be constant $S \propto V$; if V be constant $S \propto T$; but if neither be constant $S \propto TV$.

Let s, v, t be any other velocity, space and time; and let X be the space described with the velocity v in the time T: then

$S : X :: V : v$, because T is the same in both,

$X : s :: T : t$, because v is the same in both.

Therefore (Art. 50)

$S : s :: TV : tv$; that is, $S \propto TV$.

(12). *Of Arithmetical Progression.*

58. Quantities are said to be in arithmetical progression, when they increase or decrease by a common difference.

Thus 1, 3, 5, 7, 9, &c., where the increase is by the difference 2;

a, $a + b$, $a + 2b$, $a + 3b$, &c., where the increase is by the difference b;

$9a + 7x$, $8a + 6x$, $7a + 5x$, &c., where the decrease is by the difference $a + x$;

are in arithmetical progression.

59. To find any term of an arithmetical progression, multiply the difference by the number of the term *minus* one, and add the product to the first term,

if the progression be an increasing one, or subtract the product, if a decreasing one.

Thus the 10^{th} term of 1, 3, 5, &c. is $1 + 9 \times 2 = 19$.

The n^{th} term of a, $a + b$, $a + 2b$, &c. is $a + \overline{n - 1}b$.

The 6^{th} term of $9a + 7x$, $8a + 6x$, &c. is

$$9a + 7x - 5(a + x) = 9a + 7x - 5a - 5x = 4a + 2x.$$

60. To find the sum of an arithmetical progression, multiply the sum of the first and last terms by half the number of terms.

Thus the sum of 10 terms of 1, 3, 5, &c. is

$$(1 + 19) \times 5 = 100.$$

For if $1 + 3 + 5 + \&c.$ to 19 (10 terms) $= s$,

$19 + 17 + 15 + \&c.$ to 1 (10 terms) $= s$;

therefore $20 + 20 + 20 + \&c.$ to 20 (10 terms) $= 2s$,

or $20 \times 10 = 2s$, or $20 \times 5 = s$.

Also n terms of a, $a + b$, $a + 2b$, &c.

$$= (2a + \overline{n - 1}b)\frac{n}{2}.$$

For if $a + (a + b) + (a + 2b) + \&c.$

to $a + \overline{n - 1}b$ (n terms) $= s$

$(a + \overline{n - 1}b) + (a + \overline{n - 2}b) + \&c.$

to a (n terms) $= s$;

therefore $(2a + \overline{n - 1}b) + (2a + \overline{n - 1}b) + \&c.$

(n terms) $= 2s$;

therefore $(2a + \overline{n - 1}b) \times n = 2s$

and $(2a + \overline{n - 1}b) \times \frac{n}{2} = s$.

(13). *Of Geometrical Progression.*

61. Quantities are said to be in geometrical progression, or continual proportion, when the first is to the second as the second to the third, and as the third to the fourth, &c.

Or when every succeeding term is a certain multiple or part of the preceding term.

Thus 8, 12, 18, 27 are in continued proportion or in geometric progression. In this case the terms are

$$8, \quad 8 \times \frac{3}{2}, \quad 8 \times \frac{3}{2} \times \frac{3}{2}, \quad 8 \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2},$$

$$\text{or } 8, \quad 8 \times \frac{3}{2}, \quad 8 \times \left(\frac{3}{2}\right)^2, \quad 8 \times \left(\frac{3}{2}\right)^3.$$

In like manner, a, ar, ar^2, ar^3 are in geometric progression.

62. The multiplier by which each term is obtained from the preceding is called the *common ratio.*

63. To find any term of a geometrical progression, multiply the first term by that power of the common difference which has for its exponent the number of the term *minus* one.

Thus the 5th term of the progression 8, 12, 18, &c. is,

$$8\left(\frac{3}{2}\right)^4 = 8 \times \frac{81}{16} = \frac{81}{2} = 40\tfrac{1}{2}.$$

And the n^{th} term of a, ar, ar^2, &c. is ar^{n-1}.

64. To find the sum of an increasing geometrical progression, multiply the last term by the common ratio, subtract from the product the first term, and divide the remainder by the excess of the common ratio above unity.

Thus, the sum of 5 terms of 8, 12, 18, &c. is

$$\frac{\frac{81}{2} \times \frac{3}{2} - 8}{\frac{3}{2} - 1} = \frac{243 - 32}{4} \div \frac{1}{2} = \frac{211}{2} = 105\tfrac{1}{2}.$$

And the sum of n terms of a, ar, ar^2, &c. is

$$\frac{ar^{n-1} \times r - a}{r - 1} = \frac{ar^n - a}{r - 1}.$$

For if $a + ar + ar^2 +$ &c.

$+ ar^{n-1}$ (n terms) $= s$,

multiplying by r, $ar + ar^2 +$ &c.

$+ ar^{n-1} + ar^n$ (n terms) $= rs$,

and subtracting, $ar^n - a = rs - s = (r - 1)\, s$,

whence $\dfrac{ar^n - a}{r - 1}$

GEOMETRY.

Elements of Geometry. Euclid, Books *i, *ii, *iii, iv.

Book v. **Definition of Proportion.*

The first of four magnitudes is said to have the same ratio to the second which the third has to the fourth when—any *equi*-multiples *whatsoever* of the *first* and *third* being taken, and any equi-multiples whatsoever of the *second* and *fourth*,—if the multiple of the first be *less* than that of the second, the multiple of the third is also *less* than that of the fourth; or if the multiple of the first be *equal* to the multiple of the second, the multiple of the third is also *equal* to that of the fourth; or if the multiple of the first be *greater*

than that of the second, the multiple of the third is also *greater* than that of the fourth.

Ratio is the relation of quantities in respect of proportion, so that if a, b, c, d be proportional, the ratio of a to b is equal to the ratio of c to d.

*LEMMA 1. If magnitudes be proportionals according to the algebraical definition of proportion, they are also proportionals according to the geometrical definition.

If magnitudes a, b, c, d be proportionals algebraically, $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{ma}{nb} = \frac{mc}{nd}$, where ma, mc are any equi-multiples whatsoever of a, c, and nb, nd, any equi-multiples whatsoever of b, d; and if ma be less than nb, mc is less than nd; and if equal, equal; and if greater, greater. (Art. 38.) Therefore the magnitudes a, b, c, d are proportionals according to the geometrical definition.

LEMMA 2. If magnitudes be proportionals according to the geometrical definition, they are also proportionals according to the algebraical definition.

If $a : b :: c : d$ according to the geometrical definition, suppose, first, a to be any multiple, part, or parts of b, so that $a = \frac{n}{m} b$; therefore $ma = nb$; therefore by the definition $mc = nd$; therefore $\frac{c}{d} = \frac{n}{m}$; but $\frac{a}{b} = \frac{n}{m}$; therefore $\frac{a}{b} = \frac{c}{d}$.

Hence $\frac{a}{b} = \frac{c}{d}$, whenever a is any multiple, part, or parts of b. But when the quantities a, b, c, d are determined by any geometrical conditions, the fractions

$\frac{a}{b}$ and $\frac{c}{d}$ will be equal or unequal according to those conditions, and the algebraical equation will express the results of these conditions generally, without regard to magnitude. Therefore the equality cannot depend upon that particular magnitude of a or b, which makes a some multiple, part, or parts of b. Therefore, since, for those magnitudes of a and b for which a is a multiple, part, or parts of b, $\frac{a}{b}$ is equal to $\frac{c}{d}$, these fractions must be equal without any such restriction, and we shall have in all cases $\frac{a}{b} = \frac{c}{d}$.

Hence when quantities have been proved to be geometrically proportional, we may apply to them all those results of algebraical proportion which have been already proved, in Arts. 38 to 50.

EUCLID, Book VI.

Definition 1. The altitude of any figure is the straight line drawn from the vertex perpendicular to the base.

Def. 2. Similar rectilineal figures are those which have their several angles respectively equal, and the sides about the equal angles respectively proportionals.

*Prop. i. Triangles and parallelograms of the same altitude are to one another as their bases.

*Prop. ii. If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or those produced, proportionally; and if the sides, or the sides produced, be cut propor-

tionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.

*Prop. iii. If the angle of a triangle be bisected by a straight line cutting the base, the segments of the base shall have the same ratio as the other sides of the triangle; and if the segments of the base are to each other as the other sides of the triangle, the straight line drawn from the vertex to the point of section, bisects the vertical angle.

Prop. A. If the exterior angle of a triangle, made by producing one of its sides, be bisected by a straight line, which also cuts the base produced; the segments between the dividing line and the extremities of the base are to each other as the other sides of the triangle; and if the segments of the base produced are to each other as the other sides of the triangle, the straight line draw from the vertex to the point of section divides the exterior angle of the triangle into two equal angles.

*Prop. iv. The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides; that is, are the antecedents or consequents of the ratios.

Cor. to Prop. iv. Since it has been shewn (Lemma 2) that when quantities are proportionals geometrically, they are proportionals algebraically; all the consequences which are proved of algebraical proportion (Arts. 37 to 50) may be asserted of the proportionals in Props. i, ii, iii, A, iv of this Book vi.

EUCLID, Book XI.

Def. 1. A straight line is perpendicular or at right angles to a plane, when it makes right angles with every straight line meeting it in that plane.

Def. 2. A plane is parallel to another plane when they do not meet, though both are indefinitely produced.

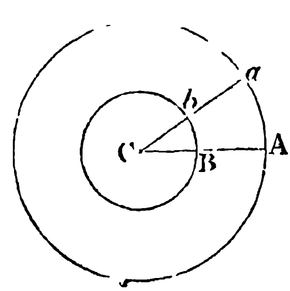
Def. 3. A plane is parallel to a straight line when they do not meet, though both are indefinitely produced.

Def. 4. A *prism* is a solid figure contained by two parallel planes, and by a number of other planes all parallel to one straight line, and cutting the first two planes so as to form polygons.

The first two planes are called the *ends* or *bases* of the prism, and the intermediate portion of the straight line to which all the other planes are parallel is the *length* of the prism.

The following Lemmas will be taken for granted: (straight lines, surfaces and solids being measured by numbers.)

Lemma 3. The arcs which subtend equal angles at the centers of two circles are as the radii of the circles.



Let the two circles be placed so that their centers coincide at C: and so that one of the lines CA containing the angle ACa in one of the circles coincides with the corresponding line CB in the other circle. Then the other lines containing the angles, namely Ca, Cb, will coincide; and it will be true that $Aa : Bb :: CA : CB$.

LEMMA 4. The area of a rectangle is equal to the product of the two sides.

If A, B be the two sides, the rectangle is $= A \times B$.

COR. If B be the base and A the altitude of a triangle, the area of the triangle is $= \frac{1}{2} A \times B$.

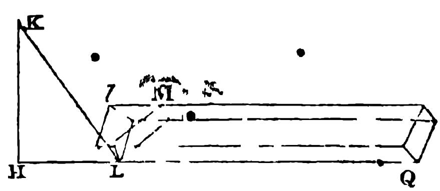
LEMMA 5. If a prism be cut by planes perpendicular to its length at different points, the areas of the sections are all similar and equal.

LEMMA 6. The solid content of a prism is equal to the product of its length and of the area of a section perpendicular to the length.

If A be the area of the section and H the length, the solid content is $= A \times H$. In this case, solid contents are measured by the number of times they contain a unit of solid content.

COR. In a uniform prism the weight is as the solid content; hence the weight of any portion of a uniform prism is proportional to its length.

LEMMA 7. If a prism be cut by two planes passing through any point of its length, one of the planes being perpendicular to the length and the other oblique to it;



and if a line be drawn at the point, perpendicular to the oblique section and intercepted by a line perpendicular to the length; the oblique section is to the perpendicular section as the portion of the perpendicular line intercepted is to the portion of the length intercepted.

Let Ll, LM be the perpendicular and the oblique section of the prism, of which the length is QL, LK perpendicular to the section LM, and KH perpendicular to the length QL. Then area LM : area Ll :: KL : HL.

MECHANICS.

BOOK I. STATICS.

DEFINITIONS AND FUNDAMENTAL NOTIONS.

1. MECHANICS is the science which treats of the laws of the motion and rest of bodies.

2. Any cause which moves or tends to move a body, or which changes or tends to change its motion, is called FORCE.

3. BODY or MATTER is any thing extended, and possessing the power of resisting the action of force.

A *rigid* body is one in which the force applied at one part of the body is transferred to another part, the relative positions of the parts of the body not being capable of any change.

4. All bodies within our observation fall or tend to fall to the earth: and the force which they exert in consequence of this tendency, is called their WEIGHT.

5. Forces may produce either rest or motion in bodies. When forces produce rest, they *balance* each other; they are in *equilibrium*; they *destroy* each other's effects.

6. STATICS is the science which treats of the laws of forces in equilibrium.

7. Two directly opposite forces which balance each other are *equal*.

Forces are directly opposite when they act in the same straight line in opposite directions.

8. Forces are capable of *addition*. Thus, when two men pull at a string in the same direction, their forces are added; and when two heavy bodies are put in the same vessel suspended by a string, their weights are added, and are supported by the string.

9. A force is *twice* as great as a given force, when it is the sum of two others, each equal to the given force; a force is *three* times as great, when it is the sum of three such forces; and so on.

10. Forces (in Statics) may be *measured* by the weights which they would support.

11. The *Quantities of Matter* of bodies are measured by the proportion of their mechanical effect.

12. The quantities of matter of two bodies are *as their weight* at the same place.

12. The *Density* of a body is measured by the quantity of matter contained in a given space.

SECTION I.

THE LEVER.

DEFINITIONS.

1. A LEVER is a rigid rod, moveable, in one plane, about a point, which is called the *fulcrum* or *centre of motion*, by means of forces which tend to turn it round the fulcrum.

2. The portions of the rod between the fulcrum and the points where the forces are applied, are called the *arms*.

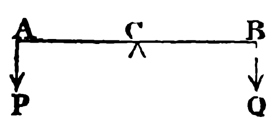
3. When the arms are two portions of the same straight line, the lever is called a *straight* lever; otherwise it is called a *bent* lever.

4. The lever is supposed to be without weight, unless the contrary be expressed.

AXIOMS.

1. If two equal forces act perpendicularly at the extremities of equal arms of a straight lever to turn it opposite ways, they will keep each other in equilibrium.

If $AC = BC$, and P and Q be two equal forces acting perpendicularly on CA and CB at A and B, they will balance each other.

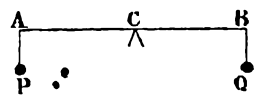


2. If forces keep each other in equilibrium, and if any force be added to one of them, it will preponderate.

COR. Hence the converse of Axiom 1 is true; if two forces P, Q acting perpendicularly at equal arms balance, they are equal. For if they are unequal, let $P + X = Q$; then $P + X$ will balance Q, by Axiom 1; but since P balances Q, $P + X$ will preponderate by Axiom 2: which is absurd. Therefore $P = Q$.

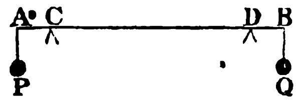
3. If two equal weights balance each other upon a horizontal straight lever, the pressure upon the fulcrum is equal to the sum of the weights, whatever be the length of the lever.

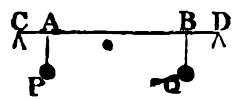
If P, Q be two equal weights which balance each other upon the horizontal lever AB, the pressure upon C is $P + Q$.



COR. If two equal forces acting perpendicularly on the arm of a straight lever balance, the pressure on the fulcrum is equal to the sum of the forces. For (Def. 10) all statical forces are equal to the weights which they would support; and hence, if for the weights, be substituted the forces which would support them, the pressure on the fulcrum is not altered.

4. If two equal weights be supported upon a straight lever on two fulcrums at equal distances from the weights, the pressures upon the two fulcrums are together equal to the sum of the weights.



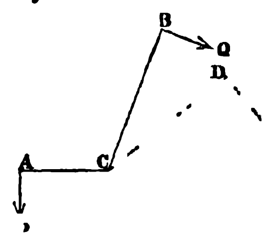


If P, Q be two equal weights which are supported upon the line AB on two fulcrums C, D, so that AC, BD are equal; the pressures upon C, D are together equal to the sum of the weights $P + Q$.

5. On the same suppositions, the pressures on the two fulcrums are equal.

6. If a force act perpendicularly on the straight arm of a bent lever at its extremity, the effect to turn the lever round the fulcrum will be the same, whatever be the angle which the arm makes with the other arm, so long as the length is the same.

If a force Q act perpendicularly on CB at its extremity B, C being the fulcrum, and an equal force R act perpendicularly on an equal arm CD, at its extremity, the effect to turn the lever round C in the two cases is equal.



7. When a force acts upon a rigid body it will produce the same effect to urge the body in the line of its own direction, at whatever point of the direction it acts.

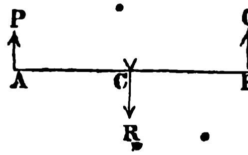
8. If a body which is moveable about an axis be acted upon by two equal forces, in two planes perpendicular to the axis, the forces being perpendicular at the extremities of two straight arms of equal length from the axis; the two forces will produce equal effects to turn the body, at whatever points the arms meet the axis.

9. If a stretched string pass freely round a fixed body, so that the direction of the string is altered, any force exerted at one extremity of the string will produce at the other extremity the same effect as if the force had acted directly.

10. If in a system which is in equilibrium, there be substituted for the force acting at any point, an immoveable fulcrum at that point, the equilibrium will not be disturbed.

11. If in a system which is in equilibrium there be substituted for an immoveable point or fulcrum the force which the fulcrum exerts, the equilibrium will not be disturbed.

Cor. If a weight be supported on a horizontal rod by two forces acting vertically at equal distances from the weight, the forces are equal to each other, and their sum is equal to the weight.



For let two forces P, Q balance each other, acting perpendicularly on the equal arms of a lever AB: then by Cor. to Ax. 2, they are equal. Also by Cor. to Ax. 3, the pressure upon the fulcrum is equal to the sum of the forces P, Q. Hence by Ax. 11, if instead of a fulcrum, there be a force R, acting at C perpendicular to the lever, and equal to the sum of P and Q, this force will balance the pressure at C, just as the fulcrum does, and there will be an equilibrium; that is, a vertical force, or weight R, will be supported by two forces equal to P, Q, acting vertically at equal distances CA, CB; and the weight R is equal to $P + Q$.

12. A perfectly hard and smooth surface, acted on at any point by any force, exerts a reaction which is perpendicular to the surface at that point; and if the surface be supposed to be immoveable, the force will be supported, whatever be its magnitude.

13. A heavy material straight line, prism, or cylinder, of uniform density, may be supposed to be composed of a row of heavy points of equal weight, uniformly distributed along the line.

14. A heavy material plane of uniform density may be supposed to be composed of a collection of

parallel straight lines of equal density, uniformly distributed along the plane.

15. A heavy solid body of uniform density may be supposed to be composed of a collection of particles, the weight of each of which is as the portion of the body which it occupies; and which particles may be considered as heavy points.

POSTULATES.

1. A prism or cylinder of uniform density, and of given length, may be taken, which is equal to any given weight.

2. A force may be taken equal to the excess of a greater given force over a less.

3. A force may be taken in a given ratio to a given force.

REMARKS ON THE AXIOMS OF STATICS.

1 The Axioms of Statics in the preceding pages are simply stated, without addition or explanation; in the same manner in which the Axioms of Geometry are stated in Treatises on Geometry. As the Axioms of Geometry are derived from the idea of space, so the Axioms of Statics are derived from the idea of *statical force* or *pressure*, and the idea of *body* or *matter*, as that which receives and transmits pressure. The student must possess distinctly this idea of force acting upon body, and body sustaining force;—of body resisting the action of force, and while it resists, transmitting this action;—of body with this mechanical property, existing in the various forms of rigid straight line, lever, plane, solid, flexible line, flexible surface;—and when he has this distinct possession of these elementary ideas, the truth of the Axioms of Statics will be seen as self-evident, and he will be in a condition to go on with the reasonings by which the following Propositions are established.

But we may make a few Remarks tending to illustrate the self-evident character of the above Axioms.

2. We shall begin with the consideration of the First Axiom of Statics (see p. 28); which is, "If two equal forces act perpendicularly at the extremities of equal arms of a straight line to turn it opposite ways, they will keep each other in equilibrium." This is often, and properly, further confirmed, by observing that there is no reason why one of the forces should preponderate rather than the other, and that, as both cannot preponderate, neither will do so. All the circumstances on which the result (equilibrium or preponderance) can depend, are equal on the two sides;—equal arms, equal angles, equal forces. If the forces are not in equilibrium, *which* will preponderate? No answer can be given, because there is no circumstance left by which either can be distinguished.

3. The argument which we have just used, is often applicable, and may be expressed by the formula, "there is no reason why one of the two opposite cases should occur, which is not equally valid for the other; and as both cannot occur (for they are opposite cases) neither will occur." This argument is called "the principle of sufficient reason;" it puts in a general form the considerations on which several of our axioms depend; and to persons who are accustomed to such generality, it may make their truth more clear.

The same principle might be applied to other cases, for example, to Axiom 6, that the effect produced on a bent lever does not depend on the direction of the arm. For if we suppose two forces acting perpendicularly on two *equal* arms of a bent lever to turn it opposite ways, these forces will balance, whatever be the angle which they make, since there is no reason why either should preponderate: it thus appears, that the force which, acting at A, would be balanced by Q in the figure to Axiom 6, would also be balanced by R, and therefore these two forces produce the same effect; which is what the axiom asserts.

4. The same reasoning might be applied to Axiom 8; for if two equal forces act at right angles at equal arms, in planes perpendicular to the axis of a rigid body, and tend to turn it opposite ways, they will balance each other, since all the conditions are the same for both forces.

5. Nearly the same may be said of Axiom 9;—if a stretched string pass freely round a fixed body, equal forces acting at its two ends will balance each other; for if it pass with perfect freedom, its passing round the point cannot give an advantage to either force. Therefore the force which will be balanced by the string at its second extremity is exactly equal to the force which acts at its first extremity. The same principle may be applied to prove Ax. 5.

6 The axioms which are perhaps least obvious are Axioms 3 and 4; for instance, the former;—that "the pressure upon the fulcrum is equal to the sum of the weights." Yet this becomes evident when we consider it steadily. It will then be seen that we conceive pressure or weight as something which must be supported; so that the whole support must be equal to the whole pressure. The two weights which act upon the lever must be somehow balanced and counteracted, and the length of the lever cannot at all remove or alter this necessity. Their pressure will be the same as if the two arms of the lever were shortened till the weights coincided at the fulcrum; but in this case, it is clear that the pressure on the fulcrum would be equal to the sum of the weights: therefore it will be so in every other case.

7. This principle, that in cases of statical equilibrium, a force is necessarily supported by an equal force, is sometimes expressed as an Axiom, by saying that "Action is always accompanied by an equal and opposite Re-action." This principle thus stated may be considered as an expression of the conception of *equality* as applied to forces; or as a Definition of *equal forces*. This principle is implied in the conception of any comparison of forces; for equilibrium and addition of forces are modes in which forces are compared, as superposition and addition of spaces are modes in which geometrical quantities are compared.

We may further observe, that this fundamental conception of *action* and *re-action* is equivalent to the conception of *force* and *matter*, which are ideas necessarily connected and correlative. Matter, as stated in page 26, is that which can resist the action of force. In Mechanics at least, we know matter only as the subject on which force acts.

8. But matter not only receives, it also transmits the action of force; and it is impossible to reason respecting the mechanical results of such transmission, without laying down the fundamental

principle by which it operates. And this accordingly is the purpose of Axioms 6, 7, 8, 9, 10, 12. When the body is supposed to be perfectly rigid, it transmits force without any change or yielding This rigidity of a body is contemplated under different aspects, in the Axioms just referred to. In Axiom 7, it is the rigidity of a rod pushed endways; in Axiom 6, the rigidity of a plane turned about a fixed point; in Axiom 8, the rigidity of a solid twisted about an axis. Axiom 9 defines the manner in which a flexible string transmits pressure, and in like manner we shall have Axioms in Hydrostatics, defining the manner in which a fluid transmits pressure. We may call Axioms 6, 7, 8, collectively, the Definition of a *rigid body* The place of these principles in our reasoning will not be thereby altered; nor will the necessity of their being accompanied by distinct mechanical conceptions be superseded.

9. Axioms 13, 14, 15, of the Statics, are all included in the general consideration, that material bodies may be supposed to consist of material parts, and that the weight of the whole is equal to the weight of all the parts; but they are stated separately, because they are used separately, and because they are at least as evident in these more particular cases as they are in the more general form.

By considerations of this nature it appears, that the axioms, as above stated, are evident in their nature, in virtue of the conceptions which we necessarily form, in order to reason upon mechanical subjects.

10. Some persons may be surprised to find the Axioms of Mechanics represented as so numerous; especially if they look for analogy to Geometry, where the necessary axioms are confessedly few, and according to some writers, none; and they may be led to think that many of the axioms here given must be superfluous, by observing that in most mechanical works the fundamental principles are stated as much fewer than these. But very few of those which are here stated are superfluous *in effect*. From the very circumstance that they *are* axioms, they are assented to when they are adduced in the reasoning, whether they have been before asserted or not; but to make our reasoning formally correct every proposition which is assumed should be previously stated. And when we consider carefully, we see that the various modifications and combinations of the ideas of

force, body, and equilibrium, along with the ideas of space of one, two, or three dimensions, readily branch out into as many heads as appear in this part of the present work.

11. Some persons may be disposed at first to say, that our knowledge of such elementary truths as are stated in the Axioms of Statics and Hydrostatics, is collected *from observation and experience.* But in refutation of this we may remark, that we cannot experimentally verify these elementary truths, without assuming other principles, which require proof as much as these do. If, for instance, Archimedes had wished to ascertain by trial whether two equal weights at the equal arms of a lever would balance each other, how could he know that the weights *were* equal, by any more simple criterion than that they *did* balance? But in fact, it is perfectly certain that of the thousands of persons who from the time of Archimedes to the present day have studied Statics as a mathematical science, a very few have received or required any confirmation of his axioms from experiment; and those who have needed such help have undoubtedly been those in whom the apprehension of the real nature and force of the evidence of the subject was most obscure.

12. We do not assert that the axioms as stated in this Treatise are given in the only exact form; or that they may not be improved, simplified, and reduced in number. But it does not seem likely that this can be done to any great extent, consistently with the rigour of deductive proof. The Fourth Axiom of Statics is one which attempts have been made to supersede: for example, Lagrange* has endeavoured to deduce it from the preceding ones. But it will be found that his proof, if distinctly stated, involves some such axiom as this:—that "If two forces, acting at the extremities of a straight line, and a single force, acting at an intermediate point of the straight line, produce the same effect to turn a body about another line, the two forces produce at the intermediate point an effect equal to the single force." And though this axiom may be self-evident, it will hardly be considered as more simple than that which it replaces.

13. Thus, the science of Statics, like Geometry, rests upon axioms which are neither derived directly from experience, nor capable of being superseded by definitions, nor by simpler prin-

* *Mécanique Analytique.* Introduction.

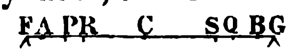
ciples. In this science, as in Geometry, the evidence of those fundamental truths resides in those convictions to which an attentive and steady consideration of the subject necessarily leads us. The axioms with regard to pressures, action, and re-action, equilibrium and preponderance, rigid and flexible bodies, result necessarily from the conceptions which are involved in all exact reasoning on such matters. The axioms do not flow *from* the definitions, but they flow irresistibly *along with* the definitions, from the distinctness of our ideas upon the subjects thus brought into view. These axioms are not arbitrary assumptions, nor selected hypotheses; but truths which we must see to be necessarily and universally true, before we can reason on to any thing else; and in Mechanics, as in Geometry, the capacity of seeing that they are thus true, is required in the student, in order that he and the writer may be able to proceed together.

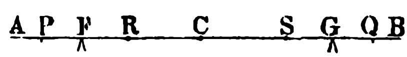
PROPOSITIONS.

N. B. The Propositions required by the University for the degree of B. A. are those which are marked by numbers; and the Enunciations are printed in larger type.

Prop. I. A horizontal prism or cylinder of uniform density will produce the same effect by its weight as if it were collected at its middle point.

Let AB be the prism or cylinder, and C its middle point. Let P, R be any points in AC, and let CQ be taken equal to CP, and CS equal to CR.





The half AC of the prism may (by Ax. 13.) be supposed to be made up of small equal weights, distributed along the whole of the line AC, as at P, R; and the half BC may in like manner be conceived to be made up of small equal weights distributed along BC; as at Q, S; of which the weight at Q is equal to the weight at P, that at S to that at R, and so on.

Let F be a fulcrum about which the prism AB tends to turn by its weight. In CB, produced if necessary, take CG equal to CF, and suppose a fulcrum placed at G.

Let the weights at P, Q, R, S be denoted by P, Q, R, S.

The two weights P and Q produce upon the fulcrums F and G pressures which together are equal to the sum of the weights $P + Q$, (Ax. 4,) or to the double

of P, since P and Q are equal. But the pressure upon each of these fulcrums is equal, (Ax. 5,) hence the pressure upon each of them is P; therefore the pressure upon the fulcrum G, arising from the two weights P and Q, is P; in like manner the pressure upon the fulcrum G, arising from R and S, is R; and so of the rest: and the whole pressure on G, arising from the whole prism AB, is the sum of all the weights P, R, &c. from A to C; that is, it is half the weight of the prism.

But if the whole prism be collected in its middle point C, the pressure upon the two fulcrums F and G will be the whole weight of the prism, and the pressures on the two fulcrums are equal; by Cor. to Ax. 11. Therefore, in this case also, the pressure on the fulcrum G is equal to half the weight of the prism. Therefore the prism, when collected at its middle point, produces the same pressure on the fulcrum G as it did before.

Therefore, when a uniform prism is collected at its middle point, it produces the same effect by its weight as it did before. Q.E.D.

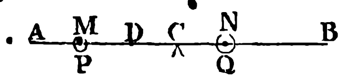
Cor. 1. A uniform prism or cylinder will balance itself upon its middle point.

Cor. 2. When a prism or cylinder thus balances upon its middle point, the pressure upon the fulcrum is equal to the weight of the prism.

Prop. II. If two weights acting perpendicularly at the extremities of the arms of a [horizontal] straight lever on opposite sides of the fulcrum balance each other, they are inversely as their distances from the fulcrum, and the pressure on the fulcrum is equal to their sum.

Let P, Q be the two weights.

Let there be a uniform prism of the length AB, equal in weight to $P + Q$ (Post. 1), and let $AD : DB :: P : Q$. Therefore, componendo, $AD + BD : AD :: P + Q : P$. But $AD + BD$ is equal in weight to $P + Q$, and the prism is uniform; therefore by Cor. to Lemma 6, the prism AD is equal in weight to P. In like manner the prism BD is equal in weight to Q.



Let C be the middle point of AB; M, the middle point of AD; N, the middle point of DB. By Prop. I. Cor. 1 and 2, the prism AB will balance on the point C, and the pressure on that point will be equal to the weight of the prism, that is to $P + Q$.

But by Prop. I. the prism AD will produce the same effect as if it be collected at its middle point M; that is, the same effect as the weight P at M. And in like manner the prism DB will produce the same effect as the weight Q at N. Therefore the whole prism AB will produce the same effect as the weight P at M, and the weight Q at N; that is, the weight P at M, and Q at N will balance on C.

But since MD is half AD, and DN is half DB, the sum MN is half the sum AB, and is therefore equal to AC. Hence taking away the common part MC, the remainder CN is equal to AM, or MD. And to MD and CN adding the common part DC, MC is equal to DN.

Now $P : Q :: AD : DB$ by construction; that is,

$$P : Q :: 2MD : 2DN; \text{ or } :: MD : DN;$$

hence, by what has been proved,

$$P : Q :: CN : MC.$$

Therefore the weights P, Q are inversely as their

distances from the point C on which they balance.
Q. E. D.

Also the weights P, Q collected at M, N produce the same effect on the fulcrum C as the prisms AD, DB; that is, as the prism AB; that is, they produce a pressure $P + Q$, as has been shewn. Q. E. D.

COR. If two *forces* acting perpendicularly on a straight lever on opposite sides of the fulcrum balance each other, they are inversely as their distances from the fulcrum, and the pressure on the fulcrum is equal to the sum of the forces.

For any forces may be represented by weights; and what is true of the weights is true of the forces.

PROP. A. If two weights acting perpendicularly at the extremities of the arms of a straight horizontal lever on opposite sides of the fulcrum are inversely as their distances from the fulcrum, they will balance each other.

As in the last Proposition, let there be a uniform prism AB, equal in weight to the sum of the weights $P + Q$; and let it be divided in D, so that $AD : DB :: P : Q$; then, as before, AD is equal in weight to P, and BD to Q.

Let M be the middle point of AD; N, of DB. And let C be a point, such that $CN : MC :: P : Q$.

$$\text{Then } CN : MC :: AD : DB$$
$$:: 2MD : 2DN :: MD : DN,$$
$$\text{whence } MC + CN : CN :: MD + DN : MD,$$

and the first and third are equal; therefore CN is equal to MD.

Hence adding DC to both, MC is equal to DN or NB; and hence AM and MC together are equal to CN and NB together; that is, AC is equal to CB; and C is the middle point of AB.

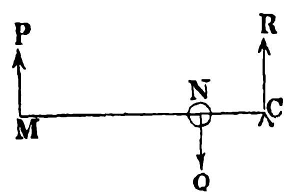
Therefore the prism AB will balance on C; and by Prop. I. if the part AD, that is, P, be collected at M, and the part DB, that is, Q, be collected at N, the effect will still be the same; that is, P and Q will balance on C. Therefore, &c. Q. E. D.

COR. 1. In this case also the pressure upon the fulcrum C is equal to $P + Q$.

COR. 2. If for weights be put any forces, the lever being in any position, the same proposition is true.

PROP. III. If two forces acting perpendicularly on a straight lever in opposite directions and on the same side of the fulcrum balance each other, they are inversely as their distances from the fulcrum; and the pressure on the fulcrum is equal to the difference of the forces.

Let MCN be the lever on which the two forces P and Q acting perpendicularly at M and N in opposite directions balance each other. Let R be a force such that $P + R$ is equal to Q, and let MNC be supposed to be a lever on which two forces P, R, acting perpendicularly at M, C on opposite sides of the fulcrum, balance each other.



Then, by Prop. II. the pressure upon the fulcrum N is equal to $P + R$, that is, to Q, and is in the direction of the forces P and R. Hence if a force $P + R$, that is Q, act perpendicularly to the lever MC at N in the direction opposite to P and R, it will supply the place of the fulcrum, and the forces, P, Q, R will still balance each other by Ax. 11. And if we place an immoveable fulcrum at C, it will supply the place of the

force R, and the forces P, Q, will still balance each other by Ax. 10.

But since P, R balance on the lever MNC, we have by Prop. II.

$$R : P :: MN : NC \text{; and therefore}$$
$$R + P : P :: MN + NC : NC \text{; that is}$$
$$Q : P :: MC : NC \text{;}$$

the forces P, Q are inversely as their distances from the fulcrum C.

Also the pressure on the fulcrum C, which replaces the force R is equal to the force R, that is to the difference of the forces P and Q. Q. E. D.

Prop. IV. To explain the different kinds of levers.

When material levers are used, the two forces which have been spoken of, as balancing each other upon the lever, are exemplified by the weight to be raised or the resistance to be overcome, as the one force, and the pressure, weight, or force of any kind, employed for the purpose, as the other force. The former of these forces is called *the Weight*, the latter is called *the Power*.

The preceding Propositions give the proportion of the Power and Weight in the case of equilibrium, that is, when the weight is not raised, but only supported; or when the resistance is not overcome, but only neutralized. But knowing the Power which will produce equilibrium with the weight, we know that any additional force will make the Power preponderate. (Ax. 2.)

Straight levers are divided into three kinds, according to the position of the Power and Weight.

1. The Lever of the First kind is that in which

the Power and Weight are on opposite sides of the Fulcrum, as in Proposition II. and A.

We have an example of a lever of this kind, when a bar is used to raise a heavy stone by pressing down one end of the bar with the hand, so as to raise the stone with the other end: the Power is the force of the hand, the Fulcrum is the obstacle on which the bar rests, the Weight is the weight of the stone.

We have an example of a double lever of this kind in a pair of pincers used for holding or cutting; the Power is the force of the hand or hands at the handle, the Weight is the resistance overcome by the pinching edges of the instrument, the Fulcrum is the pin on which the two pieces of the instrument move.

2. The Lever of the Second kind is that in which the Power and the Weight are on the same side of the Fulcrum, the Weight being the nearer to the Fulcrum.

We have an example of a lever of this kind, when a bar is used to raise a heavy stone by raising one end of the bar with the hand, while the other end rests on the ground, and the stone is raised by an intermediate part of the bar. The Fulcrum is the ground, the Power is the force exerted by the hand, the Weight is the weight of the stone.

We have an example of a double lever of this kind in a pair of nutcrackers. The Power is the force of the hand exerted at the handles; the Weight is the force with which the nut resists crushing; the Fulcrum is the pin which connects the two pieces of the instrument.

3. The Lever of the Third kind is that in which the Power and the Weight are on the same side of the fulcrum, and the Weight is the further from the fulcrum.

In this kind of lever, the Power must be greater

than the Weight in order to produce equilibrium, by Prop. III. Therefore by the use of such a lever, force is lost. The advantage gained by the lever is, that the force exerted produces its effect at an increased distance from the fulcrum.

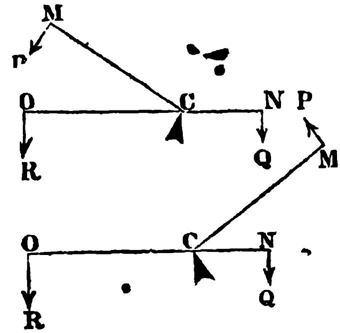
We have an example of a lever of this kind in the anatomy of the fore-arm of a man, when he raises a load with it, turning at the elbow. The elbow is the Fulcrum, the Power is the force of the muscle which, coming from the upper arm is inserted into the fore-arm near the elbow, the Weight is the load raised.

We have an example of a double lever of this kind in a pair of tongs used to hold a coal. The Fulcrum is the pin on which the two parts of the instrument turn, the Power is the force of the fingers, the Weight is the pressure exerted by the coal upon the ends of the tongs.

PROP. V. If two forces acting perpendicularly at the extremities of the arms of any lever balance each other, they are inversely as the arms.

Let MCN be any lever: and let P, Q acting perpendicularly on the arms CM, CN balance each other; then $P : Q :: CN : CM$.

Produce NC to O, taking CO equal to CM; and at O let a force R equal to P act perpendicularly on the lever NCO, to turn it in the same direction as P. Then since CM is equal to CO, and therefore P to the force R, both acting perpendicularly to the arms, by Axiom 6, P and R will produce the same effect to turn the lever round the fulcrum C; and therefore since P balances Q, R will balance Q.

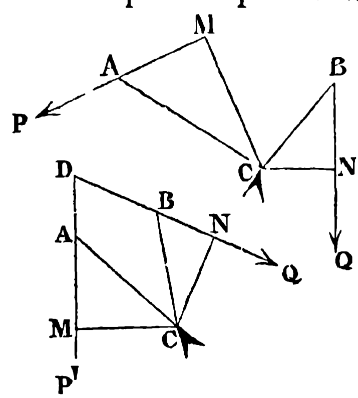


But since forces R and Q balance on the straight lever OCN, by Prop. II. $R : Q :: CN : CO$; and since P is equal to R, and CO to CM, $P : Q :: CN : CM$; or the forces P, Q are inversely as their arms. Q.E.D.

Prop. VI. If two forces acting at any angles on the arms of any lever balance each other, they are inversely as the perpendiculars drawn from the fulcrum to the directions in which the forces act.

Let ACB be the lever on which the forces P, Q acting at any angles balance each other; and let CM, CN be the perpendiculars from the fulcrum C in the directions of the forces; then $P : Q :: CN : CM$.

The lever ACB is supposed to be rigid, so that the arms AC, BC cannot alter their respective positions. Hence we may suppose the plane ACB to be a rigid indefinite plane, moveable about the point C, and AC, BC to be lines in this plane. Therefore the forces P, Q, which act at the points A, B, will by Axiom 7, produce the same effect as if they act at the points M, N respectively: therefore if they act at these points M, N they will still balance.



Hence by Prop. V. $P : Q :: CN : CM$; or the forces are inversely as the perpendiculars CM, CN. Q.E.D.

Cor. 1. The converse is true, that if $P : Q :: CN : CM$, the forces will balance.

Cor. 2. If P, Q, CM, CN be expressed in num-

bers when P, Q balance, $P \times CM = Q \times CN$; and when $P \times CM = Q \times CN$, P and Q balance.

Definition of the moment of a force. If lines be expressed in numbers, the product which arises when a force acting on a lever is multiplied by the perpendicular from the fulcrum of the lever upon the direction of the force is called the *moment* of the force.

It appears by the last Corollary, that when two forces balance on a lever, their moments are equal; and when their moments are equal they balance.

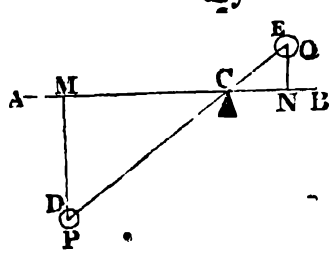
Also if the moment of one force be the greater, that force will preponderate.

Cor. 3. If X be any force acting on the lever ACB, and CO the perpendicular upon its direction, and if $X \times CO = P \times CM$, the force X will produce upon the lever the same effect as P. For $X \times CO = Q \times CN$; therefore, by this Proposition, X will balance Q; which is what P does.

Cor. 4. If the two forces P, Q act at the same point D, the proposition is still true.

Prop. VII. If two weights balance each other on a straight lever when it is horizontal, they will balance each other in every position of the lever.

Let it be supposed that the weights P, Q, acting at A, B, balance each other upon the lever when it is in the horizontal position ACB; the weights P, Q will balance each other upon the same lever in any other position, as DCE.



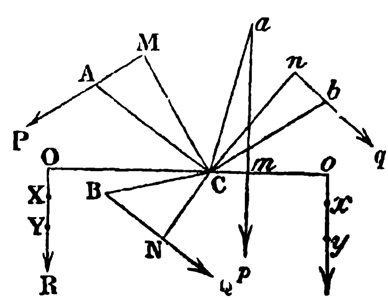
Draw DM, EN vertical, meeting the horizontal line ACB. Then, in the triangles DCM, ECN, the vertical angles DCM, ECN are equal; and DMC,

ENC are equal, being right angles; therefore the remaining angles of the triangles are equal, and the triangles are equiangular and similar. Therefore *DC* : *CM* :: *EC* : *CN*, and alternately *DC* : *EC* :: *CM* : *CN*. But since *P*, *Q* balance each other on *AB*, *Q* : *P* :: *AC* : *CB*; and *AC* is equal to *DC*, and *CB* to *EC*, because *ACB* and *DCE* are the same lever; therefore *Q* : *P* :: *DC* : *EC*; therefore by what precedes, *Q* : *P* :: *CM* : *CN*; therefore, by Prop. VI. the weights *P*, *Q*, acting at the points *D*, *E*, will balance each other. Q.E.D.

Cor. The pressure upon the fulcrum *C* in every position of the lever *DE* is equal to the sum of the weights *P* and *Q*. For in every position the effect of the weights *P*, *Q* is the same as if they acted at *M*, *N*, by Axiom 7. But in this case, by Prop. II. the pressure on the fulcrum *C* is the sum of the weights.

Prop. B. If any number of forces act upon a lever, and tend to turn it opposite ways, and if the sum of the moments of the forces which tend to turn the lever one way be equal to the sum of the moments of the forces which tend to turn it the other way, the forces will balance each other.

Let the forces *P*, *Q*, *R*, tend to turn the lever one way, and let *CM*, *CN*, *CO* be the perpendiculars on their directions; and let the forces *p*, *q*, tend to turn the lever the other way, and let *Cm*, *Cn* be the perpendiculars on their directions; and let $P \times CM + Q \times CN + R \times CO$ be equal to $p \times Cm + q \times Cn$; the forces will balance each other.



Let any two lines CO, Co be taken, and let forces act at O and o, perpendicularly to CO, Co, to turn the lever opposite ways, namely, at O, a force X, such that $CO : CM :: P : X$, by Post. 3. that is, such that $X \times CO = P \times CM$; and also a force Y, such that $Y \times CO = Q \times CN$, and a force R; and also at o, a force x, such that $x \times Co = p \times Cm$, and a force y, such that $y \times Co = q \times Cn$.

Then, by Cor. 3 to Prop. VI, the force X will produce the same effect as the force P, and the force Y will produce the same effect as the force Q; and therefore the forces P, Q, R will produce the same effect as X, Y, R acting at O. In like manner the forces p, q will produce the same effect as x, y, acting at o.

But the forces X, Y, R, acting at O, will balance the forces x, y, acting at o, if $(X + Y + R) \times CO$ be equal to $(x + y) \times Co$, by Prop. VI; that is, if $X \times CO + Y \times CO + R \times CO$ be equal to $x \times Co + y \times Co$; that is, by the construction, if $P \times CM + Q \times CN + R \times CO$ be equal to $p \times Cm + q \times Cn$. Therefore, &c. Q.E.D.

Cor. 1. If the forces be weights acting on a straight horizontal lever, the same is true, putting for the perpendiculars on the directions of the forces, the portions of the lever CM, CN, &c. intercepted between the fulcrum and the weights. (See next figure).

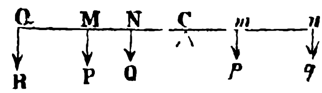
Cor. 2. The converse of this Proposition and of Cor. 1 are true.

Prop. C. If any forces act perpendicularly upon a lever, the pressure on the fulcrum is equal to the sum of the forces.

It will first be proved that if any number of forces acting perpendicularly upon a lever balance each other, they may be separated into parts, so that, retaining

their positions, they form pairs, each of which pairs would balance on the fulcrum separately.

Let P, Q, R, p, q be any forces which balance each other on the lever OM $NCmn$. If each force on one side of the fulcrum has its moment equal to that of a corresponding force on the other side, it is clear that each force will balance the corresponding one on the other side, and the forces are already in such pairs as are mentioned above. But if not, let any moment on one side, as $P \times CM$, be less than a moment on the other side, as $p \times Cm$. Assume a force u such that $Cm : CM :: P : u$, by Post. 3: therefore $P \times CM = u \times Cm$; therefore $u \times Cm$ is less than $p \times Cm$, and u is less than p; let $p = u + x$. Then if p be s parated into parts u and x, the pair P and u will balance each other separately, because their moments are equal.



In the same manner, of the forces Q, R, x, q, take any other as Q, of which the moment $Q \times CN$ is less than the moment of $q \times Cn$ of a force q on the other side of the fulcrum. Assume a force v such, that $Cn : CN :: Q : v$, therefore $Q \times CN = v \times Cn$; and let $q = v + y$. Then if q be separated into v and y, the pair Q and v will balance each other separately, for the same reason as before.

And of the forces R, x, y, the moment $x \times Cm$ must be less than $R \times CO$. Assume $X \times CO = x \times Cm$; and let $R = X + Y$. The pair X, x will balance each other separately, as before.

But because the forces P, Q, R, p, q balance on the lever, it follows (by Cor. 2 to Prop. B) that

$$P \times CM + Q \times CN + R \times CO = p \times Cm + q \times Cn;$$

and hence, since

$$R = X + Y, \text{ and } p = u + x, \text{ and } q = v + y,$$
$$P \times CM + Q \times CN + X \times CO + Y \times CO$$
$$= u \times Cm + x \times Cm + v \times Cn + y \times Cn;$$

and it has been supposed that

$$P \times CM = u \times Cm, \text{ and } Q \times CN = v \times Cn,$$
$$\text{and } X \times CO = x \times Cm;$$

hence the remainder

$$Y \times CO \text{ is} = y \times Cn;$$

and the pair Y, y will balance each other.

Therefore the forces have been separated into pairs,

$$P, u; \quad Q, v; \quad X, x; \quad Y, y;$$

which balance each other separately.

Also it is plain that the same proof may be applied in any case; for at each step the number of forces which are not in pairs is diminished by one; and therefore the reduction may always be effected by as many steps as there are forces, wanting one.

Hence the Proposition is manifest; for the pressure upon the fulcrum arising from each pair is equal to the sum of the two forces of that pair (Prop. II); therefore the whole pressure is equal to the sum of all the pairs; that is, to the sum of all the forces.

SECTION II.

COMPOSITION AND RESOLUTION OF FORCES.

DEFINITIONS.

1. WHEN two forces act at the same point, they produce the same statical effect as a certain single force, acting at that point. This single force is called the *resultant* of the two; they are called its *components*. The two forces produce the single force by being *compounded*, and it may be *resolved* into the two.

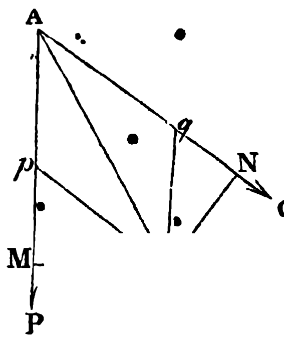
2. Straight lines may *represent* forces in direction and magnitude, when they are taken in the direction of the forces and proportional to their magnitude. When forces are so represented, if *AB* represent any force, *BA* represents an equal and opposite force. A force represented by any line, as *AB*, is often called "the force *AB*."

3. Forces may be *represented* by lines parallel to them in direction and proportional to them in magnitude.

PROP. VIII. If the adjacent sides of a parallelogram represent the component forces in direction and magnitude, the diagonal will represent the resultant force in direction and magnitude.

The proof will consist of two parts; for the direction, and for the magnitude.

First, the diagonal will represent the resultant force in *direction*.



Let Ap, Aq represent in magnitude and direction the forces P, Q, acting at A; complete the parallelogram $ApCq$; and draw AC; draw also CM, CN perpendicular upon Ap, Aq.

The triangles CpM, CqN have right angles at M and N, and the angles MpC, CqN are equal, each being equal to MAN; therefore the triangles CpM, CqN are equiangular and similar. Therefore $CM : CN :: Cp : Cq$; that is, $CM : CN :: Aq : Ap$. But Ap, Aq represent the forces P, Q in magnitude; therefore $CM : CN :: Q : P$. Therefore, by Prop. VI, Cor. 4, if the forces P, Q act on the plane PAQ, supposed to be moveable about the point C, they will balance each other, producing a pressure on the fulcrum C.

Therefore the single force which produces the same effect as P, Q will produce a pressure upon the point C, but will not turn the plane about C. But this cannot be the case except the single force act in the line AC; for if it acted in any other direction, a perpendicular might be drawn from C upon the direction, and the force would produce motion, by Axiom 2. Therefore the resultant acts in the direction AC.

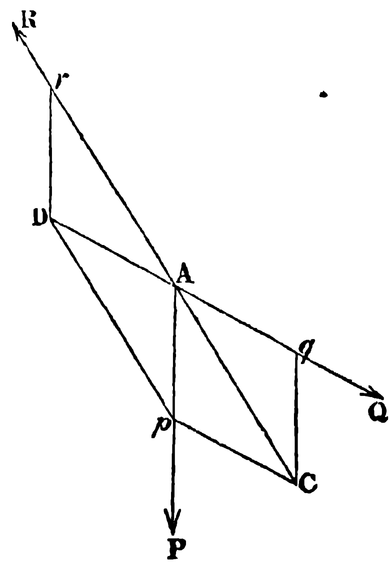
Hence if a point, acted upon by two forces Ap, Aq, be kept at rest by a third force, this force must act in the direction CA. For otherwise it would not balance the force in the direction AC, to which the forces Ap, Aq are equivalent.

Hence also if three forces act on a point, and keep each other in equilibrium, each of them is in the direction of the diagonal of the parallelogram whose sides represent the other two.

Secondly, the diagonal will represent the resultant force in *magnitude.*

By the proof of the former part the two forces Ap, Aq will be kept in equilibrium by a force in the direction CA. Let Ar represent this force in magnitude. Therefore the three forces Ap, Aq, Ar keep each other in equilibrium. Complete the parallelogram $ApDr$, and draw its diagonal DA. Then by the proof of the former part, the force Aq is in the direction DA; and therefore DAq is a straight line.

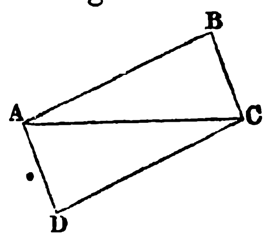
Hence in the triangles CAq, DAr, the vertical angles CAq, DAr are equal; and Cq, Dr are parallel to each other, because Cq and Dr are both parallel to Ap; and Cr meets them; therefore the angle qCA is equal to the alternate angle DrA. Therefore the triangles CAq, DAr are equiangular. Also Cq and Dr are equal, for each is equal to Ap, being opposite sides of parallelograms pq. pr. Therefore (Euc. VI. 8) the other sides of the triangles CAq, DAr are equal; therefore CA is equal to Ar. But Ar represents in magnitude the force which keeps in equilibrium Ap, Aq; and since Ar acting in the opposite direction would balance Ar, the force which produces



the same effect as Ap, Aq, is Ar acting in the opposite direction. Therefore AC, which is equal to Ar, represents in magnitude the force which produces the same effect as Ap, Aq; that is, the resultant of Ap, Aq.

Hence, if the components be represented in magnitude and direction by the sides of a parallelogram, the resultant is represented in magnitude and direction by the diagonal of the parallelogram. Q. E. D.

PROP. IX. If three forces represented in magnitude and direction by the sides of a triangle taken in order, act on a point, they will keep it in equilibrium.



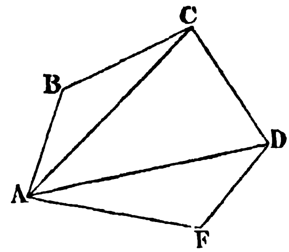
Let three forces, represented in magnitude and direction by the three lines AB, BC, CA, act on the point A, they will keep it in equilibrium. Complete the parallelogram $ABCD$, then the force which is represented by BC is also represented by AD, (Def. 3 of this Sect.) and acts at the point A. And the resultant of the forces AB, AD is represented in magnitude and direction by AC (Prop. VIII); therefore the forces AB, BC produce the same effect as AC; and therefore the forces AB, BC, CA produce the same effect as AC, CA; that is, they will keep the point A in equilibrium.

COR. 1. If three forces which keep a point in equilibrium be in the direction of three lines forming a triangle, they are proportional to those lines.

COR. 2. Any two forces AB, BC, which act at a point A, are equivalent to a force AC.

PROP. D. If any number of forces, represented in magnitude and direction by the sides of a polygon taken in order, act on a point, they will keep it in equilibrium.

Let forces AB, BC, CD, DE, EA act upon a point A; they will keep it in equilibrium. By Prop. IX, Cor. 2, the forces AB, BC are equivalent to a force AC; therefore the forces AB, BC, CD are equivalent to the forces AC, CD; that is, by the same corollary, to a force AD. Therefore again, the forces AB, BC, CD, DE are equivalent to the forces AD, DE; that is, again by the same corollary, to a force AE. Therefore, finally, the forces AB, BE, CD, DE, EA are equivalent to forces AE, EA, and therefore will keep the point A in equilibrium.



SECTION III.

MECHANICAL POWERS.

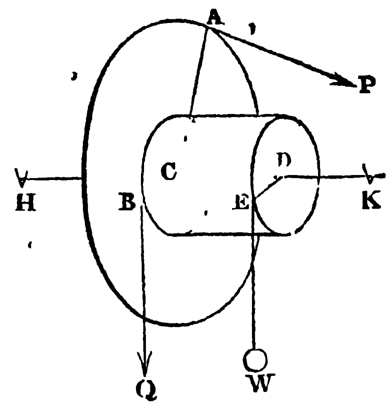
THE WHEEL AND AXLE.

DEF. *The Wheel and Axle* is a rigid machine, which is moveable about an axis, and on which two forces, tending to turn it opposite ways, act in two planes perpendicular to the axis; the one force (the *Power*) acting by means of a string stretched and wrapt on the circumference of a circle perpendicular to the axis, called the *Wheel*; the other force (the *Weight*) acting by means of a string wrapt on the surface of a cylinder having the axis of motion for its axis, and called the *Axle*.

PROP. X. There is an equilibrium upon the wheel and axle, when the power is to the weight as the radius of the axle to the radius of the wheel.

Let AB be the wheel, and DEB the axle, the whole being moveable about the axis $HCDK$; the power P, acting at A, perpendicular to CA, the radius of the wheel; and the weight W, acting at E, perpendicular to DE, the radius of the axle. Also let $P : W :: DE : CA$; then there will be an equilibrium.

In the plane of the wheel AB, let CB be drawn



from the axis, equal to *DE* the radius of the axle; and let a force *Q*, equal to *W*, act at *B* perpendicular to *CB*. Then, by Axiom 8, the two forces *Q*, *W* produce equal effects in turning the machine. But the force *Q* will balance *P*, by Prop. VI, Cor. 1, because $P : W :: DE : CA$, and therefore $P : Q :: CB : CA$, *Q* being equal to *W*, and *CB* to *DE*· therefore *W* will balance *P*, and there will be an equilibrium. Q.E.D.

Cor. 1. On the wheel and axle when there is equilibrium, the moments of the power and weight are equal.

Cor. 2. If the power and weight do not act perpendicularly to the radii of the wheel and axle, it will appear, by the reasoning of Prop. VI, that there will still be an equilibrium if their moments are equal.

Cor. 3. If several forces acting upon a body moveable about a fixed axis, and acting in planes perpendicular to the axis, tend to turn it opposite ways, there will be an equilibrium when the sum of the moments of the forces which tend to turn the body one way is equal to the sum of the moments of the forces which tend to turn the body the other way. This may be proved by reasoning similar to that of Prop. B.

Cor. 4. If a heavy body be moveable about any axis, it will be in equilibrium when the moments of the weights of the two parts into which it is divided by a vertical plane passing through the axis, are equal: for these two parts will tend to turn it opposite ways.

In this case, the moment of each particle of the body is found by drawing from the particle a vertical line meeting a horizontal line which is perpendicular to the axis. The length of this perpendicular, measured from the vertical to the axis, multiplied into the weight of the particle, is the moment of the particle, if the axis is horizontal; and is proportional to the moment if the axis be in any other position.

Cor. 5. Conversely, if these moments are not equal, there cannot be equilibrium.

THE PULLEY.

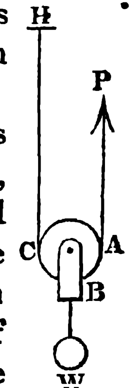
Def. A *Pulley* is a machine in which one part, (the *Block*) being stationary, a stretched string can pass freely round another part, (the *Sheave*).

A pulley is *fixed* when the block is fixed, and *moveable* when the block is moveable.

The Power is the force which acts at the string; the Weight is the weight supported.

Prop. XI. In the single moveable pulley, where the strings are parallel, there is an equilibrium when the power is to the weight as 1 to 2.

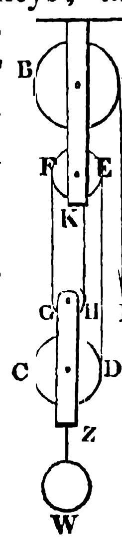
Let ABC represent a pulley in which B is the block, AC the sheave, and in which the strings PA, HC are parallel: there is an equilibrium when $P : W :: 1 : 2$.



By Axiom 9, since the string passes freely round the sheave AC, the force P, which is exerted on the string PA, is equal to that which the string CH exerts on the fixed point H; and therefore the reaction which the fixed point H exerts by means of the string HC, is also equal to P. And the two forces, each equal to P, which act by means of

the parallel strings *AP*, *CH*, may be considered as balancing each other upon a lever *AC*, the fulcrum of which is in the point of the block *B*, by which the weight *W* is supported. Therefore by Prop. II, the pressure on the fulcrum is the sum of these forces, that is, it is the double of *P*; and this pressure on the fulcrum of the block *B* is balanced by the pressure or weight of *W* upon the block in the opposite direction, in the case of equilibrium; therefore, in the case of equilibrium, *W* is double of *P*, or *P* : *W* :: 1 : 2.

Prop. XII. In a system in which the same string passes round any number of pulleys, and the parts of it between the pulleys are parallel, there is an equilibrium when power (*P*) : weight (*W*) :: 1 : the number of strings at the lower block.

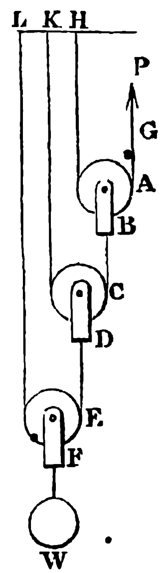


Let *AC* represent the system of pulleys; the string *ABCDEFGHK* passing round all the pulleys, and the portions *CB*, *DE*, *GF* *HK*, being all parallel. By Axiom 9, the forces exerted by each of these strings will be equal to *P*; therefore the forces which they exert upon the lower block will each be equal to *P*. And these forces may be considered as acting upon a lever, the fulcrum of which is in the point of the block *Z*, by which the weight *W* is supported. Therefore by Prop. C, the pressure upon this fulcrum is equal to the sum of the forces of the strings, that is, it is as many times *P* as there are strings at the lower block. And this pressure on the fulcrum in the lower block is balanced by the pressure or weight of *W* in the

opposite direction in the case of equilibrium; therefore in the case of equilibrium, $P : W :: 1$: number of strings in the lower block. Q. E. D.

PROP. XIII. In a system in which each pulley hangs by a separate string, and the strings are parallel, there is an equilibrium when $P : W :: 1$: that power of 2 whose index is the number of moveable pulleys.

Let AL represent the system of pulleys; each pulley A, C, E hanging by a separate string, and the strings being all parallel. It appears by the reasoning of Prop. XI, that



P : force of string BC :: 1 : 2;
force of string BC : force of string DE :: 1 : 2;
force of string DE force of string FW :: 1 : 2.

And there will be as many such proportions as there are moveable pulleys A, C, E. Also in compounding these proportions the proportion compounded of the former ratios in each proportion will be P : force of string FW; and the proportion compounded of the latter ratios in each proportion will be 1 : 2 raised to that power whose index is the number of ratios. Therefore

P : force of string FW :: 1 : 2 raised to that power. And the force of the string FW is equal to the weight W, because it supports it in the case of equilibrium. Therefore, &c. Q. E. D.

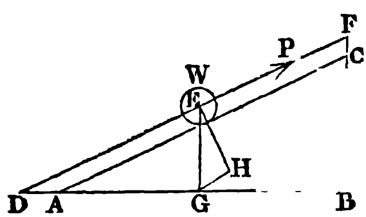
THE INCLINED PLANE.

DEF. *The Inclined Plane*, when spoken of as a mechanical power, is a plane supposed to be perfectly

smooth and hard. The inclined plane is represented by a line drawn in a vertical plane, and is supposed to pass through this line and to be perpendicular to the vertical plane. A vertical line is supposed to be draw in the vertical plane from the upper extremity of the inclined plane; and both this vertical line, and the line which represents the inclined plane, are cut by a horizontal line or *base*, drawn in the same vertical plane. The portion of the inclined line and of the vertical line intercepted between the upper point of the plane and its horizontal base, are the *length* and the *height* of the inclined plane respectively.

PROP. XIV. The weight (W) being on an inclined plane, and the force (P) acting parallel to the plane, there is an equilibrium when $P : W ::$ the height of the plane : its length.

Let AC be an inclined plane of which AC is the length, and let W be a weight on the inclined plane supported by a force P, acting in the direction EF parallel to AC.



The force of the weight W acts in a vertical direction; draw EG vertical to represent this force. Also draw EH perpendicular and GH parallel to the plane AC.

The force EG is equivalent to the two forces EH, HG, (Prop. IX, Cor. 2); of these, the force EH is balanced by the reaction of the plane AC, which will balance any force perpendicular to AC, by Axiom 12; and the weight W will be kept at rest, if the force HG be counteracted by an equal and opposite force P, acting in the direction EF.

Therefore there will be equilibrium if P be represented by GH, when W is represented by EG; that is, $P : W :: GH : EG$.

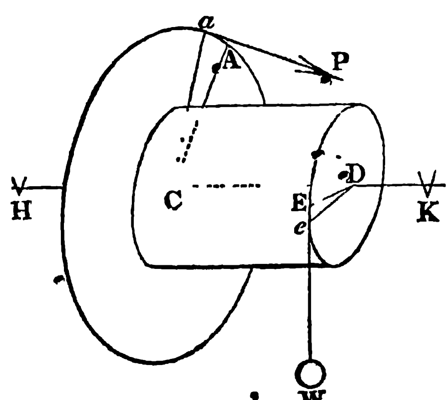
But since EH is perpendicular and GH parallel to the plane AC, EHG is a right angle and therefore equal to ABC. Also the angle EGH is, by parallels, equal to GED, that is, to BFD, that is, to BCA. Therefore the two triangles ABC, EHG, have two angles equal, each to each, and are therefore equiangular, and therefore also similar. Hence $GH : EG :: BC : AC$, and therefore, by what has been proved already, $P : W :: BC : AC$, that is, $P : W ::$ height of plane : length of plane. Therefore, &c. Q.E.D.

VELOCITY.

DEF. If two points pass through certain spaces respectively in the same time, the *Velocities* of the two points are to each other in the proportion of these two spaces.

PROP. XV. If P and W balance each other on the wheel and axle, and the whole be put in motion, $P : W :: W$'s velocity : P's velocity.

The construction being the same as in Prop. X, let the machine turn round its axle CD through an angle ACa, or EDe; so that the radius of the wheel at which the power acted, moves out of the position Ca into the position CA; and so that the radius of the axle at which the



power acted, moves out of the position De into the position DE. Then the string by which the power P acts will be unwrapt from the portion aA of the circumference of the wheel, and therefore P will move through a space equal to aA. Also in the same time the string at which W acts will be wrapt upon the axle by a space equal to eE, and therefore W will move through a space equal to eE. Therefore by the definition of velocity, aA, eE are as the velocities of P and W.

But since the wheel and axle is a rigid body, turning about the axis CD, all the parts move in planes perpendicular to the axis, and turn through the same angle; and since the plane of the wheel ACa, and of the axle EDe are both perpendicular to the axis, the angles ACa, EDe are the angles through which the radii CA, DE turn. Therefore the angles ACa, EDe, at the centers of the circles ACa, EDe are equal; and therefore, by the Lemma 3, $DE : CA :: Ee : Aa$.

But by Prop. X, $DE : CA :: P : W$; and by what has been just shewn, $Ee : Aa ::$ W's velocity : P's velocity; therefore $P : W ::$ W's velocity : P's velocity. Q.E.D.

PROP. XVI. To shew that if P and W balance each other in the machines described in Propositions XI, XII, XIII, and XIV, and the whole be put in motion, $P : W ::$ W's velocity in the direction of gravity : P's velocity.

Part first: proof for the systems of pulleys described in Propositions XI, XII, XIII.

In Prop. XI, if W be raised through any space, as one inch, the string on each side of the pulley A will be liberated for one inch; and therefore P will

be at liberty to descend two inches; therefore W's velocity : P's velocity :: 1 : 2; and since by Prop. XI, $P : W :: 1 : 2$, $P : W ::$ W's velocity : P's velocity.

In Prop. XII, if W be raised through any space, as one inch, each string at the lower block will be liberated one inch, and therefore as many inches of string will be liberated as there are strings at the lower block; and P will be at liberty to descend through a space equal to the whole of this. Therefore the space described by W : space described by P :: 1 : number of strings at the lower block; and hence by Prop. XII, and by the definition of velocity, $P : W ::$ W's velocity : P's velocity.

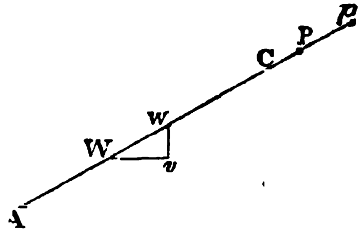
In Prop. XIII, if W be raised through any space, as one inch, each of the two strings at the lowest pulley E will be liberated one inch; therefore the pulley C will be liberated 2 inches, and will rise through 2 inches; therefore on each side the block C, 2 inches of string will be liberated; therefore the pulley A will be liberated 2×2 inches; therefore the string on each side the pulley A will be liberated 2×2 inches; therefore the string at which P acts will be liberated $2 \times 2 \times 2$ inches, and since this happens in the same time that W is liberated one inch, W's velocity : P's velocity :: $1 : 2 \times 2 \times 2$. And it is clear that the last term is that power of 2 whose index is the number of moveable pulleys.

But by Prop. XIII, $P : W :: 1 : 2 \times 2 \times 2$ as before; therefore, by what has been proved, $P : W$ :: W's velocity : P's velocity.

Part second: proof for the Inclined Plane described in Prop. XIV.

Let AC be the inclined plane, the weight W

being supported by the force P acting parallel to the plane. Let W move to w, and P to p in the same time; and draw Wv horizontal and wv vertical. Then wv is the space



through which W moves in the direction of gravity, while P moves through the space Pp, or Ww, which is equal to Pp, because the string wP is always of the same length. Therefore by the definition of velocity, W's velocity in the direction of gravity : P's velocity :: wv : Ww.

But since Wv is horizontal, or parallel to AB, and wv vertical, or parallel to CB, the triangle Wwv is similar to ACB. Therefore wv : Ww :: BC : AC, that is, wv : Ww :: height of the plane : length of the plane. But by Prop. XIV, this proportion is that of P : W; therefore by what has been proved, P : W :: W's velocity in the direction of gravity : P's velocity.

COR. In the case of the inclined plane, if the string by which W is supported pass over a point C and hang vertically, as WCQ, and if Q balance W, Q will descend through a space Qq equal to Ww, when W descends through a space Ww; and we may prove, as before, Q : W :: W's velocity in the direction of gravity : P's velocity.

SECTION IV

THE CENTER OF GRAVITY.

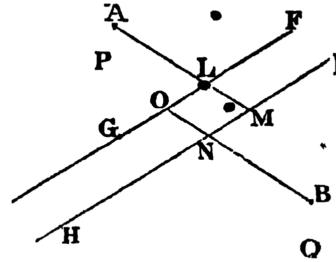
DEF. THE *Center of Gravity* of any body or system of bodies is the point about which the body or the system will balance itself in all positions.

COR. If a straight line pass through the center of gravity of a body, the body will balance itself on this line in all positions. For since the body will balance itself in all positions upon the center of gravity, if this center be supported, the body will be supported in all positions. But if the line passing through the center of gravity be supported, the center will be supported; and therefore if the line passing through the center of gravity be supported, the body will be supported in all positions; therefore it will balance itself on this line in all positions.

It is assumed that every body has a center of gravity.

PROP. XVII. If a body balance upon a straight line in all positions, the center of gravity is in that line.

Let *HK* be a line on which the system balances itself in all positions; and since every system has a center of gravity, if possible let *G*, which is not in *HK*, be the center of gravity.



Let *GF* be drawn parallel to *HK*; then, if any line in the plane *FGHK*, as *LM*, or *ON*, be perpendicular to one of these parallels, it will be perpendicular to the other. Let the body, with these lines, be turned

round the line HK, till LM is horizontal, in which case any other perpendicular, as ON, will also be horizontal. Let P be a particle, the vertical line from which meets the horizontal line ML, produced if necessary, in A; let Q be a particle, the vertical line from which meets the horizontal line ON in B; and in like manner let vertical lines be drawn from the other particles of the body, meeting horizontal lines which are perpendicular to FG and HK. Also let P, Q, be the weights of particles from which the vertical lines PA, QB are on opposite sides of the lines GF, HK.

Since the body balances on the line HK, the sum of all such moments as $P \times AM$ on the one side of the line HK must be equal to the sum of all such moments as $Q \times BN$ on the other side of the line by Prop. X, Cor. 1. And since, by the corollary to the Definition of the center of gravity, the body balances on the line GF, the sum of all such moments as $P \times AL$ on the one side of the line GF must, for the same reason. be equal to the sum of all such moments as $Q \times BO$ on the other side of the line GF.

But when we take the moments of the particles of the body with respect to the line GF, instead of HK, each of the moments on the side A, as $P \times AM$, is diminished by $P \times LM$, so as to become $P \times AL$; and each of the moments on the side B, as $Q \times BN$, is increased by $Q \times NO$, so as to become $Q \times BO$: besides which there are particles, the vertical lines from which fall between the lines HK, GF, which are on the side A of the line HK, and on the side B of the line GF; and of which the moments still further diminish the sum of the moments on the side A, and increase the sum of the moments on the side B, when we exchange the line HK for the line GF.

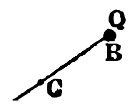
Therefore if the sums of the moments on the sides A and B of the lines HK be equal, the sums cannot

be equal when we move the line into the position GF, and therefore by Prop. X, Cor. 5, the equilibrium cannot subsist for this second line also.

Therefore the point G, out of HK, cannot be the center of gravity; and therefore the center of gravity must be in HK. Q.E.D.

PROP. XVIII. To find the center of gravity of two heavy points.

Let A, B, be the two heavy points; their weights being P and Q. Join AB; and take in AB a point C, such that $P + Q : Q :: AB : AC$; C will be the center of gravity of A, B.

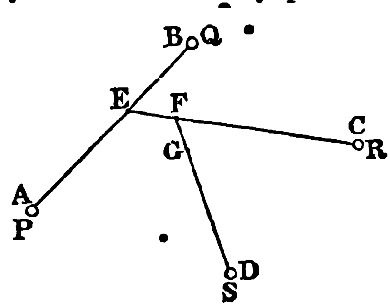


Since $P + Q : Q :: AB : AC$, by division $P : Q :: BC : AC$. Therefore by Prop. II, A and B will balance each other on the line AB in a horizontal position, because in that case the weights act perpendicularly to the lever. Therefore by Prop. VII, A, B will balance each other on C in every other position of the line AB. Therefore by the definition of the center of gravity, C is the center of gravity of the heavy points A, B.

COR. The pressure upon the center C in every position is equal to $P + Q$, by the Corollary to Prop. VII.

PROP. XIX. To find the center of gravity of any number of heavy points.

Let A, B, C, D be any number of heavy points; their weights being P, Q, R, S. Join AB, and take a point E in AB, such that $P + Q : Q :: AB : AE$; join EC, and take a point F in EC, such that $P + Q + R : R :: EC : EF$; join FD,



and take a point G in FD, such that $P+Q+R+S : S :: FD : FG$; G will be the center of gravity of P, Q, R, S.

Since $P+Q : Q :: AB : AE$, by Prop. XVIII, and Cor. E is the center of gravity of the points A, B; and in every position of AB the pressure upon E is equal to $P+Q$. But since $P+Q+R : R :: EC : EF$, by division $P+Q : R :: CF : EF$; therefore $P+Q$ at E and R at C will balance upon F when EC is horizontal by Prop. II, and when EC is in any other position by Prop. VII; and the pressure upon F in any position will be $P+Q+R$, by the Cor. to Prop. VII. Therefore in any position P, Q, R will balance upon F, and F is the center of gravity of P, Q, R.

Again, since $P+Q+R+S : S :: FD : FG$, by division, $P+Q+R : S :: DG : FG$; and $P+Q+R$ at F, and S at D, will balance in every position of FD, by Propositions II and VII. And the pressure upon G will, in every position of FD, be $P+Q+R+S$, by Cor. to Prop. VII.

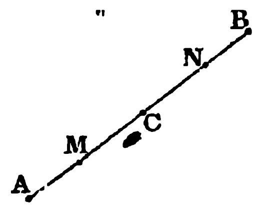
Therefore in every position of FD, EC, and BA, the points A, B, C, D will balance upon G; and therefore G is the center of gravity of A, B, C, D.

Cor. 1. It has been shewn that in every position of A, B, C, D the pressure upon G, the center of gravity, is equal to the sum of the weights.

Cor. 2. Every system of heavy points has a center of gravity; for the above construction is always possible.

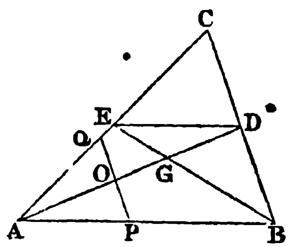
Prop. XX. To find the center of gravity of a straight line.

Let AB be the straight line; bisect it in C; C will be the center of gravity.



Take CM and CN equal, and the line may be considered as composed of pairs of equal particles, placed at points such as M, N, by Axiom 13. But the two particles at M, N balance each other upon the point C in all positions, by Prop. II and VII. And all the other pairs of particles will balance for the like reasons. Therefore the whole line will balance upon C in all positions. Therefore the point C is the center of gravity of the whole line.

Prop. XXI. To find the center of gravity of a plane triangle.



Let ABC be the triangle; bisect BC in D, and join AD; and bisect AC in E, and join BE; let G be the point of intersection of AD, BE; G is the center of gravity of the triangle.

Draw any line PQ parallel to BC, meeting AD in O; it is easily seen that the triangles AOP, ADB are similar, as also AOQ, ADC.

$$\begin{aligned} \text{Hence } OP : OA &:: DB : DA; \\ \text{and } OA : OQ &:: DA : DC; \\ \text{therefore } OP : OQ &:: DB : DC. \end{aligned}$$

But DB is equal to DC, therefore OP is equal to OQ, and O bisects PQ.

By Axiom 14, the triangle ABC may be considered as made up of straight lines PQ, parallel to BC. And the center of gravity of any one of these lines, as PQ, is at O in the line AD; therefore each of these lines will balance upon AD in any position; therefore the whole triangle, which is made up of these lines, will balance upon AD in any position, and therefore the center of gravity of the triangle is in the line AD.

In like manner, the triangle may be considered as made up of straight lines parallel to AC, and it may be proved by similar reasoning that the center of gravity of the triangle is in the line BE.

Therefore the center of gravity of the triangle is at G, the intersection of AD and BC. Q.E.D.

Cor. If we join DE, it is easily shewn that the triangles CBA, CDE are similar as also AGB, DGE,

therefore $DE : AB :: CD : CB$;

but by construction $CD : CB :: 1 : 2$;

therefore $DE : AB :: 1 : 2$.

Again $GD : AG :: DE : AB$,

therefore $GD : AG :: 1 : 2$;

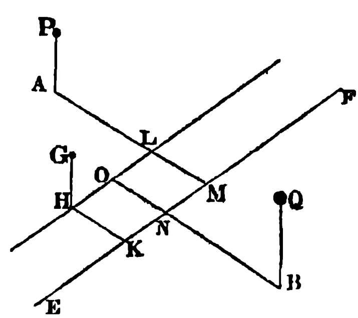
and by composition $AD : AG :: 3 : 2$;

AG is two-thirds of AD, and DG is one-third of AD.

In like manner BG, and GE, are two-thirds and one-third of BC respectively.

Prop. E. Any body will have the same effect in producing equilibrium about a given fixed line, as if it were collected at its center of gravity.

Let EF be the given fixed line, and G the center of gravity of the body. Let PA, QB vertical lines from any particles P, Q of the body, meet horizontal lines AM, BN, which are perpendicular to EF; and let GH be a vertical line which meets the horizontal line HK which is also perpendicular to EF.



The effect of the body in producing equilibrium depends upon the excess of the moments such as $P \times AM$, on one side of the line EF, above the moments

such as $Q \times BN$, on the other side of the line; and is the same so long as this excess is the same. This follows from Prop. X, Cor. 3.

Now since G is in the center of gravity, the body balances on the point G, and therefore on the line HL; for if HL be supported, G is supported. Therefore the sum of all the moments, such as $P \times AL$, on the one side, is equal to the sum of all the moments such as $Q \times BO$, on the other side. And $Q \times BO$ is equal to $Q \times BN + Q \times NO$. Therefore adding $P \times LM$ to both, the sum of moments such as $P \times AL + P \times LM$, or $P \times AM$, is equal to the sum of moments such as $Q \times BN + Q \times NO + P \times LM$. Therefore the excess of moments such as $P \times AM$ over moments such as $Q \times BN$ is the sum of moments such as $Q \times NO + P \times LM$; that is, such as $Q \times HK + P \times HK$, or $(Q + P) \times HK$; because LM and NO are each equal HK.

Now if all particles such as P and Q be transferred to G, their effect in producing equilibrium depends upon the sum of moments, such as $(P + Q) \times HK$; therefore it is the same as before.

Hence if all the particles P, Q be transferred to the center of gravity G, the effect in producing equilibrium is the same as before. But the whole body may be considered as made up of such particles; by Axiom 15. Therefore if a body be collected at its center of gravity, its effect in producing equilibrium will not be altered. Q.E.D.

COR. 1. The effect of the body to disturb equilibrium about a line will be the same as if the body were collected at its center of gravity G. For the effect to disturb equilibrium is the effect to produce equilibrium when an adequate force is applied to counteract the tendency to disturb equilibrium.

Cor. 2. The effect of a body to produce or disturb equilibrium about a point is the same as if the body were collected at the center of gravity. For any line being drawn through the point, the effect is the same about this line, by Cor. 1; and the equilibrium cannot be disturbed about a point, without being disturbed about some line passing through that point.

Note. If the fixed line be horizontal, the moment of each particle, which measures its effect in producing equilibrium, is the product of the weight of the particle multiplied by the horizontal line perpendicular to the fixed line and intercepted by a vertical line drawn from the particle.

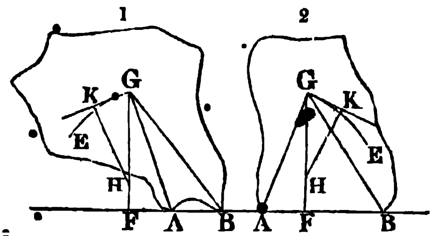
If the fixed line be not horizontal, take in it any point Z, draw ZY vertical, and YX perpendicular to the fixed line. Then the moment of any particle about the fixed line will be less than the above product in the proportion of YX to YZ. For the force arising from the weight of the particle being represented by the vertical line ZY, may be resolved into forces ZX, XY; of which ZX will not produce any effect to turn the body about the fixed line, and XY only will be effective.

Def. By the *Base* of a body is meant a side of it, touching another body, and on which its direct pressure is supported.

If the body fall over, it tends to turn round one edge of its base, whether the base slide or not.

Prop. XXII. When a body is placed upon a horizontal plane, it will stand or fall, according as the vertical line, drawn from its center of gravity, falls within or without its base.

Let $ABCD$ be the body, AB its base, G its center



of gravity. First let GF, the vertical line drawn from the center of gravity, fall upon the horizontal plane BA without the base, as at F. Take in GF any line GH to represent the weight of the body, and draw GK perpendicular to AG and HK parallel to AG.

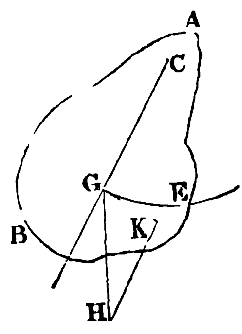
(Fig. 1) If the body fall over the edge A of the base, it will tend to turn round the edge A of the base, that is, to describe the arc GE of which the radius is AG. Now by Prop. E, the effect of the body is the same as if it were collected at the point G. Therefore the force exerted to produce this effect may be represented by the vertical line GH. And the force GH is equivalent to the forces GK, and KH, (acting at G). Of these, the force KH acts in the line GA, passing through A, and therefore produces no tendency to motion about A. But the force GK tends to make the body move in the direction GK, which is a tangent to the arc GE; and thus to make the base AB turn round the point A, quitting the plane at B. And there is no force to counteract this tendency; therefore the body will turn round the edge A, on the side on which the perpendicular GF falls.

(Fig. 2.) But if the perpendicular GH fall between A and B, as before, the effect may be represented by the vertical line GH, and the force GH is equivalent to the forces GK, KH. Of these KH (which acts at G) passes through A and does not tend to make the body turn round the edge A; but the force GK, which is a tangent to the arc GE, tends to make the body turn round A in the direction GE. But since the body is rigid, and AB is in contact with the sup-

porting plane, the body cannot turn round the point *A* in the direction *GE*, for the pressure thus produced on the horizontal plane is resisted and supported. In like manner the body cannot turn round the edge *B* by the action of the force *GH*; therefore in this case the body cannot fall.

PROP. XXIII. When a body is suspended from a fixed point, it will rest only with its center of gravity in the vertical line passing through the point of suspension.

Let *AB* be a body suspended from a fixed point *C*, and *G* its center of gravity. If *CG* be not vertical, draw *GH* vertical, and (in the vertical plane *CGH*,) *GK* perpendicular to *CG*, and *HK* parallel to *CG*. The weight of the body will produce the same effect as if it were collected at the point *G*, and may be represented by the line *GH*. But the force *GH* is equivalent to *GK*, *KH*; and of these, the force *KH* (which acts at *G*) is in the line *CG*, and is supported by the fixed point at *C*; and the force *GK* tends to make the body move in *GK*, which is a tangent to *GE*, the path in which the point *G* can move round the fixed point *C*; and there is no force to counteract this tendency, therefore the body will move in this path; and will not rest in the position *AB*.

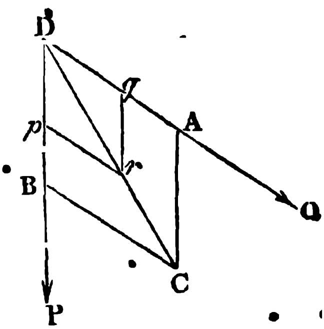


But if *CG* be vertical, the weight will be supported by the fixed point *C*, and there will be no force to produce motion; therefore the body will rest in that position.

Therefore the body will rest only when *CG* is vertical. Q.E.D.

PROP. F. If two forces tending to turn a body about a fixed point, and acting in a plane perpendicular to the axis of motion, balance each other, the pressure on the fixed point is the same as it would be if the two forces were transferred to the point retaining their direction and magnitude.

Let P, Q, be two forces, acting to turn a body about a fixed point C. Draw CA parallel to the force P, and CB parallel to the force Q; the pressure on C is the same as if the forces P, Q, acted in the lines AC, BC.



Produce the directions of the forces to meet in D, and complete the parallelogram $CADB$. The force P produces the same effect as if it acted at the point D in P's direction by Axiom 7; and similarly the force Q produces the same effect as if it acted at D. And if Dp, Dq represent the forces P, Q, and the parallelogram $Dprq$ be completed, the diagonal Dr will represent the force at D to which P and Q are equivalent. But the direction of the force Dr must pass through the point C, as in Prop. VIII, and will produce the same effect as if it acted at C; and the force Dr acting at C is equivalent to the forces qr, pr, acting in directions parallel to qr, pr, by Prop. VIII: that is, the force Dr is equivalent to the forces Dp, Dq, acting in the lines AC, BC; that is, the forces P, Q, acting in the lines BP, AQ are equivalent to forces P, Q acting in AC, BC. Therefore the pressure upon the fixed point C is the same as if the forces P, Q were transferred to that point. Q.E.D.

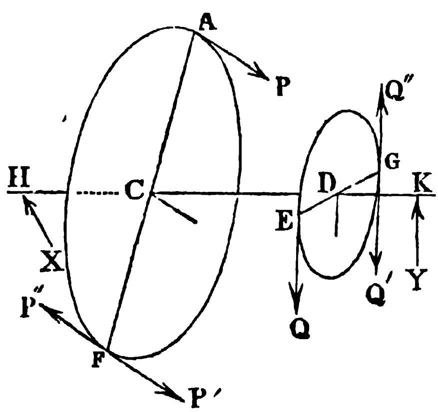
COR. 1. If, instead of the fixed point at C, we substitute the pressure which that point exerts, there

will be equilibrium by Axiom 11. Hence, if a body be acted upon by three forces in the same plane, of which one passes through the intersection of the other two, and is equal to the resultant of the other two, the body will be in equilibrium.

Cor. 2. Conversely if there be equilibrium, these conditions obtain. This follows from Axiom 2.

Prop. G. If two forces tending to turn a body round a fixed axis, and acting in two planes perpendicular to the axis, balance each other, (as in the Wheel and Axle,) the pressures upon the points of the axis where the body is supported, are the same as they would be, if the two forces, retaining their direction and magnitude, were transferred to the axis, at the points where the perpendicular planes meet it.

Let P, Q, be two forces acting perpendicularly at the arms CA, DE, to turn a body round the axis HK, the planes CAP, DEQ being perpendicular to HK; and let the forces balance. Let X, Y be the pressures exerted by the fulcrums at H and K. which pressures balance the forces P, Q. Then X and Y are the s if the forces P and Q, continuing parallel to themselves, were transferred to C and D.



Let AC be produced to F, CF being equal to CA, and at F in the plane PAC, and perpendicular to AF; let two forces P', P'', each equal to P, act in opposite directions. These forces will balance each other and

will be equivalent to no force; and therefore if the forces P', P'' are added to the system, the equilibrium will not be disturbed.

In like manner produce ED to G, DG being equal to ED, and at G, in the plane QED, and perpendicular to DG, let two forces Q', Q'', each equal to Q, act in opposite directions: these forces will not disturb the equilibrium. Therefore the six forces P, P', P'', Q, Q', Q'', acting in the manner described, will be supported by the forces X, Y; that is, the eight forces P, P', P'', Q, Q', Q'', X, Y, balance each other.

The forces P'', Q'', are situated in exactly the same manner with regard to vertical lines and planes drawn upwards, as P, Q are, with regard to vertical lines and planes drawn downwards. Therefore P'', Q'', would balance each other on the axis HK, and would produce at H and K pressures equal and opposite to those which P, Q produce. But the forces X, Y are equal and opposite to the pressures which P, Q produce, for they balance those pressures. Therefore the forces P'', Q'' produce at H, K the pressures X, Y.

The forces P, P' are equivalent to a force double of P acting at C, parallel to P; and the forces Q, Q' are equivalent to a force at D double of Q, parallel to Q.

Hence the six forces P, P', P''; Q, Q', Q'' are equivalent to X, Y, at H, K, and to $2P$, $2Q$ at C, D. And the eight forces P, P', P'', Q, Q', Q'', X, Y are equivalent to $2X$, $2Y$ at H, K, and to $2P$, $2Q$, at C, D.

But these eight forces balance each other; therefore $2X$, $2Y$, acting at H, K, balance $2P$, $2Q$, acting at C, D: and therefore X, Y, which balance P, Q, acting at A, E, would balance P, Q, acting at C, D. Q.E.D.

BOOK II.

HYDROSTATICS.

DEFINITIONS AND FUNDAMENTAL NOTIONS.

1. HYDROSTATICS is the science which treats of the laws of equilibrium and pressure of fluids.

2. Fluids are bodies the parts of which are moveable amongst each other by very small forces, and which when pressed in one part transmit the pressure to another part.

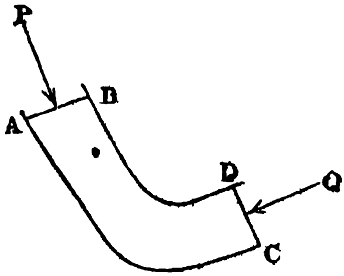
3. Some fluids are *compressible* and *elastic*; that is, they are capable of being made to occupy a smaller space by pressure applied to the boundary within which they are contained, and when thus compressed, they resist the compressing forces and exert an effort to expand themselves into a larger space. Air is such a fluid.

4. Other fluids are *incompressible* and *inelastic*; not admitting of being pressed into a smaller space nor exerting any force to occupy a larger. Water is considered as such a fluid in most hydrostatical reasonings.

5. In all fluids which have weight, the weight of the whole is composed of the sum of the weights of all the parts.

AXIOMS.

1. If a fluid of which the parts have no weight be contained in a tube of which the two ends are similar and equal planes, two equal pressures applied perpendicularly at the two ends will balance each other.



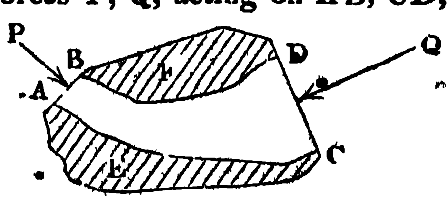
Let *ABCD* be the tube, *AB*, *CD* its two equal ends: the equal forces *P*, *Q*, acting perpendicularly on these ends will balance each other.

2. If two forces acting upon two portions of the boundary of a fluid balance each other, and if a force be added to one of them, it will prevail, and drive out the fluid at the part of the surface acted on by the other force.

Cor. Hence if *P* and *Q* in Axiom 1 balance, they are equal.

3. If a fluid be at rest in any vessel, and if any forces, acting on two portions of the boundary of the fluid, balance each other, they will also balance each other if any portions of the fluid become rigid without altering the magnitude, position, or weight of any of their parts.

Thus if the two forces *P*, *Q*, acting on *AB*, *CD*, parts of the surface of a vessel containing fluid, balance each other; they will also balance each other if the parts *E* and *F* of the fluid be supposed to become rigid, the magnitude, position and weight, of all the parts of *E*, *F*, remaining unaltered.



4—5

4. Any plane surface pressed by a fluid may be divided into any number of particles, and the pressure on the whole is equal to the sum of the pressures on each of the particles.

5. When a plane surface is pressed by a fluid, the pressure exerted on the surface, and the pressure of the surface on the fluid, are perpendicular to the plane.

6. We may reason concerning fluids supposing them to be without weight: and we shall obtain the pressures which exist in heavy fluids, if we add, to the pressures which would take place if the fluids had no weight, the pressures which arise from the weight.

7. When a finite mass of fluid is considered as consisting of small particles of any form or size, and when the consequences of our reasoning do not depend upon the magnitude of the particles, we may, in our reasoning, neglect the magnitude or weight of any single particle, and the consequences will still be true in a heavy fluid.

REMARKS ON THE AXIOMS OF HYDROSTATICS.

1. As the Axioms of Geometry are derived from the idea of space, and the Axioms of Statics from the ideas of pressure and of solid coherent matter; the Axioms of Hydrostatics are derived from the idea of pressure, and from the idea of fluid matter;—matter which, without coherence or rigidity, can still sustain pressure and transmit it in all directions; or, as we may express it more briefly, from the idea of *fluid pressure*. It is not enough to conceive a fluid as a body the parts of which are perfectly moveable: for the mere notion of mobility includes no conception of force or pressure. We must conceive fluid as transmitting pressure, in order to perceive the evidence of the Axioms of Hydrostatics.

2. The First Axiom of our Hydrostatics,—that if a fluid be contained in a tube of which the two ends are similar and equal planes acted on by equal pressures, it will be kept in equilibrium —follows from the principle of sufficient reason, for there is no reason why either pressure should preponderate. If, for example, the curvature of the tube, or any such cause, affected the pressure at either end, this condition would be a limitation of the property of transmitting pressure in all directions, and would imply imperfect fluidity; whereas the fluidity is supposed to be *perfect*

3 For the like reasons, we might assume as an *Axiom* the First *Proposition* of the Hydrostatics, that fluids transmit pressure *equally* in all directions, from one part of their boundary to the other; for if the pressure transmitted were different according to the direction, this difference might be referred to some cohesion or viscosity of the fluid; and the fluidity might be made more perfect, by conceiving the difference removed Therefore the proposition would be necessarily and evidently true of a perfect fluid.

4. But instead of laying down this as an axiom, Axiom 3 is introduced—that any part of a fluid which is in equilibrium, may be supposed to become rigid. This axiom leads immediately to Proposition I, and it is, besides, of great use in all parts of Hydrostatics.

If we had to reason concerning flexible bodies, we might conveniently and properly assume a corresponding axiom for them;—namely, that, of a flexible body which is in equilibrium, any part may be supposed to become rigid. And we might give a reason for this, by saying that rigidity implies forces which resist a tendency to change of form, when any such tendency occurs; but in a body which is in equilibrium, there is no tendency to change of form, and therefore the resisting forces vanish. It is of no consequence what forces *would* act *if there were* a stress to bend the body: since there *is not* any such stress, the rigidity is not called into play, and therefore it makes no difference whether we suppose it to exist or not.

The same kind of reasons may be given, in order to shew, what Axiom 3 asserts, the admissibility of introducing, in the case of equilibrium of a fluid, rigidity, instead of that susceptibility of change of figure, (still greater than flexibility,) which fluidity implies. Since the mass is perfectly fluid, its particles exert no

constraint on each other's motions; but then, because they are in equilibrium, no constraint is needed to keep them in their places. They are as steadily kept there (so long as the same forces continue to act) as if they were held by the insurmountable forces which connect the parts of a perfectly rigid body. We may therefore suppose the inoperative forces of rigidity to be present or absent among the particles, without altering the other forces or their relations*. And hence we see the truth of Axiom 3 of the Hydrostatics.

5. The last axiom of Hydrostatics (Ax. 7) is introduced in order that we may be able to reason concerning the quantity of fluid pressure, by supposing the fluid divided into small particles. To speak of the particles as finite would lead us into error, since they are not of any known finite magnitude; and to speak of them as indefinitely small, would involve us in the difficulties of the Higher Geometry, in which the Ideas of Limits or Differentials are introduced. The Axiom will be self-evident if we consider the particles as microscopic in magnitude, and of corresponding weight.

* This Axiom is employed familiarly by Newton and many other eminent mathematicians.

SECTION. I.

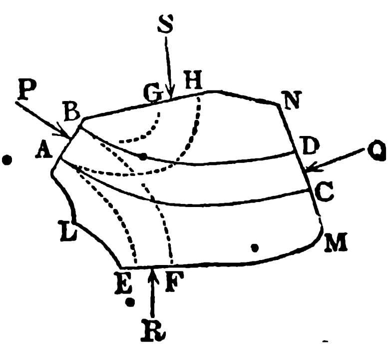
PRESSURE OF NON-ELASTIC FLUIDS.

PROPOSITIONS.

PROP. I. FLUIDS press equally in all directions.

First, a fluid at rest presses equally in all directions on equal plane portions of the vessel which contains it, if we neglect the weight of the fluid.

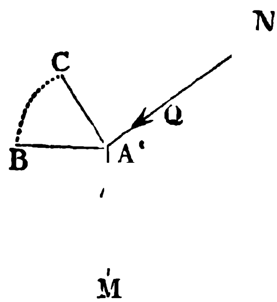
Let LMN be the close vessel, AB, CD, EF, GH, similar and equal plane portions of the surface of the vessel; let two forces P, Q acting on AB, CD, portions of the boundary of the fluid, balance each other; and let a tube $ACBD$ be imagined, passing from AB to CD. Let the portions of the fluid, ACL, BDN become rigid; then, by Axiom 3, the forces P, Q still balance each other; but by Cor. to Ax. 2, in this case the forces P, Q are equal. And in like manner it may be shewn that the forces P, R are equal, as also the forces P, S. And P, Q, R, S the forces which act on the boundary of the fluid and balance each other, are the pressures on similar and equal portions of the containing vessel. Therefore the pressures exerted on all such portions are equal.



Secondly, in a fluid at rest any particle is equally

pressed in all directions upon similar and equal plane surfaces.

Let A be any point in a fluid, and let AM, AN be any two directions. Let AB be a plane perpendicular to AM, and AC a similar and equal plane perpendicular to AN. Let the (geometrical) solid, of which the planes AB, AC are boundaries, be completed, and be considered as a particle of the fluid. And let P, Q be the forces which act on the planes AB, AC, and preserve the equilibrium. Let the whole of the fluid which surrouuds the solid ABC be supposed to become rigid: therefore, by Axiom 3, the forces P, Q still balance each other.



Let the portion BAC of fluid have no weight; therefore, by the proof of the first part, the forces P, Q are equal to each other.

But by Axiom 7, since this consequence does not depend upon the magnitude of the particle ABC, we may neglect the weight of the particle ABC, and the consequence will be true.

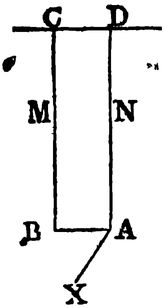
Therefore, in a fluid at rest, the pressures P, Q, which act upon a particle in the two directions MA, NA, are equal. Q.E.D.

Cor. A particle of fluid is equally pressed on any two equal and similar portions of its surface.

Prop. II. The pressure upon any particle of a [heavy] fluid of uniform density is proportional to its depth below the surface of the fluid.

First, when there is a vertical column of fluid reaching from the particle to the upper surface.

When the surface pressed is horizontal, let *AB* be the horizontal surface pressed, and *ABCD* the column reaching to the surface, the sides *AD*, *BC* being vertical.

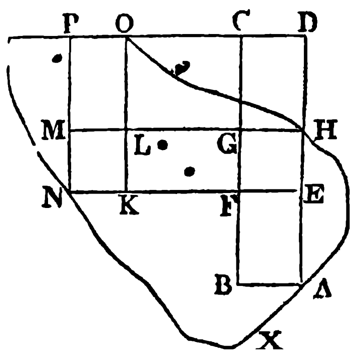


Let the column *ABCD* be divided into any number of equal particles by horizontal planes, drawn at equal vertical intervals. And each of these particles will sustain the pressure of the particle above it, and will transmit this pressure to the particle below it, by Prop. I; and will also press upon the particle below with its weight, by Ax. 6. Therefore the pressures on the particles at the distances of 1, 2, 3, &c. intervals below the surface will be as 1, 2, 3, &c.: that is, they will be as the depths.

When the surface pressed is not horizontal, let *AX* be another plane surface of the particle pressed, equal to the plane *AB*. By Prop. I, the pressure upon *AX* is equal to the pressure upon *AB*; therefore it is as the depth *AD*.

Secondly, when there is not a vertical column reaching from the particle to the upper surface.

Let *AB* be the horizontal surface pressed, *OP* the surface of the fluid, *OD* horizontal; and *AD* vertical.



Draw *AH* vertical till it meets the side of the vessel; take *HE* = *AB*, and draw *EN* horizontal till it meets the opposite side of the vessel; take *NK* = *AB*, and draw *KO* vertical; and so on if necessary; we shall in this way arrive at the upper surface of the fluid. Draw *HM*, *NP*, so as to com-

plete the zigzag tube *ABEMO* which passes from the plane *AB* to the upper surface of the fluid. Also the surfaces *GH, EF, LM*, are all equal to *AB*.

Let the column *MP*, and also the column *BC* be divided into equal particles by horizontal planes at equal vertical intervals, as in the former part of the proof. Then the pressure upon *EF* is equal to the pressure upon *GH*, together with the weight of the particle *GE*, by Axiom 6. But the pressure upon *GH* is equal to the pressure upon *ML*, by Prop. I, because *GH* is equal to *ML*. Therefore the pressure upon *EF* is the same as if a column *ED* extended to the surface: and therefore, as in the proof of the former part, the pressure on any particle in *AE* is the same as if a column *AD* extended to the surface; that is, by the former proof, it is as the depth *AD*. Q.E.D.

Cor. 1. Hence it appears that if a heavy fluid be contained in a vessel of which some parts are over the fluid, any particle of such a part exerts a pressure downward upon the fluid, equal to that which would exist if there were, instead of the particle of the vessel, a vertical column of fluid extending to the horizontal surface of the fluid.

Thus the particle of the side of the vessel which is over *GH*, presses downwards with the same force as if, instead of that particle of the vessel, there were a vertical column of fluid *GHDC*.

Cor. 2. Any portion of the side of a vessel which is over the fluid, presses downwards upon the fluid with the same force as if there were a vertical column of fluid over that part, and the side of the vessel were removed.

The part *OH* of the side of the vessel presses downwards with the same force as if the side *OH* were

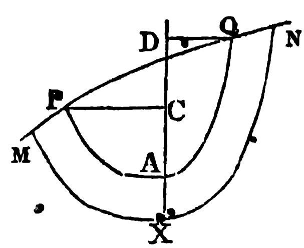
removed, and there were a column of fluid OHD over the fluid OLH.

For the pressure of the part of the side OH downwards is the sum of the pressures of each particle of OH downwards; which is, by Cor. 1, the sum of vertical columns, reaching to the horizontal surface, and standing upon each particle of OH: and the sum of these vertical columns, is a column standing on the part OH, and reaching to the surface. Therefore the whole downward pressure is equal to the whole column.

PROP. III. The upper surface of a heavy fluid of uniform density, and at rest, is horizontal.

Let PQ be the upper surface of a heavy fluid. If possible, let P, Q not be in a horizontal plane. Let A be any point in the fluid, AX the plane surface of a particle. Draw PC, QD horizontal, and ACD vertical.

By Prop. II. the pressure upon AX arising from the weight of the fluid is as AC on the side P; and for the same reason it is as AD on the side Q: and these are opposite pressures upon the plane AX. Therefore the fluid cannot be at rest except these are equal; that is, except $AC = AD$; therefore PQ is not otherwise than horizontal. Q.E.D.



PROP. IV. If a vessel, the bottom of which is horizontal, and the sides vertical, contain a heavy fluid, the pressure upon the bottom is equal to the weight of the fluid.

The pressures of the vertical sides are horizontal, and do not increase or diminish the pressure downwards. Therefore the whole weight of the fluid will be sustained in the same manner as if there were no forces acting on the sides. Let the whole fluid become rigid. Then since it is now a solid (rigid) body, the pressure upon the base is equal to the weight of the body. But by Axiom 3, the pressure is the same as before; therefore the pressure of the fluid on the base is equal to the weight. Q.E.D.

Cor. 1. The pressure of a vertical column of height H on its horizontal base B is as $B \times H$: for this is the content of the column.

Cor. 2. If AX (fig. p. 87) be a particle of the bottom which is not horizontal, the pressure on AX is as $AX \times AD$: for if $AB = AX$ be horizontal, the pressure on AB is as $AB \times AD$, by Cor. 1 : and the pressure on AX is equal to the pressure on AB, by Prop. I.

Prop. V. To construct and explain the hydrostatic paradoxes.

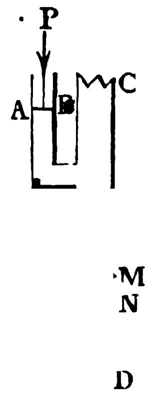
The hydrostatic paradoxes are,

1. That any pressure P, however small, may, by means of a fluid, be made to balance any other pressure W, however great.

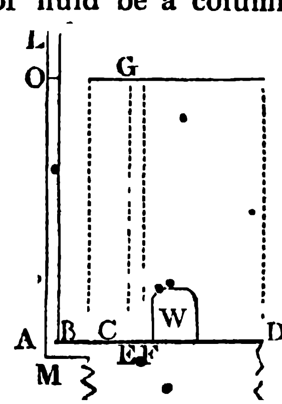
2. That any quantity of fluid, however small, may, by means of its weight, be made to balance a weight W, however great.

1. The ratio of W to P, however great, may be expressed by a number n.

Let two planes, *AB*, *CD* be taken, such that 1 : *n* :: *AB* : *CD*; and let a close machine be constructed in which these planes are moveable, so as they can exert pressure on the fluid: as, for example, if *AB* be a *piston*, or plug sliding in a tube, which enters a vessel, and if *CD* be a rigid plane closing a flexible part of the vessel, like the board of a pair of bellows; and let *P* act on *AB*, and let the fluid be in equilibrium. Then the plane *CD* may be divided into *n* surfaces, each (*MN*) equal to *AB*. By Prop. I., the pressure upon each of these surfaces is *P*, and hence the whole pressure on *CD* is (B. I. Prop. C.) the sum of all these pressures: that is, it is *n* times *P*; and if therefore *W* be *n* times *P*, *W* acting at the surface *CD* will be balanced by *P* acting at *AB*.



2. Let the given quantity of fluid be a column of which the base is *B* and the height *H*, and let the given weight *W* be equal to *n* times the weight of this column. Take a plane *CD* equal to *n* times *B*, and let a machine be constructed in which there is a vertical tube *LM*, of which the horizontal section *AB* is the surface *B*, and which enters a vessel; and let *CD* be a horizontal plane moveably connected with the vessel, as before. And let the vessel *LMND* be filled with the fluid up to the plane *CD*, and let the weight *W* be placed on the plane *CD*, and the tube *LM* be filled with fluid to the point *O* at the height *H* above *CD*, so that *ABCD* being horizontal, *AO* is equal to *H*.



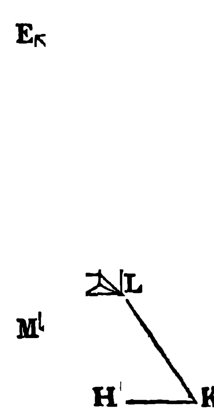
The fluid BO and the weight W will balance each other.

For the plane CD may be divided into n particles as EF, each equal to the plane B; and OG being horizontal, the pressure of the fluid upwards on each of these is equal to a column of fluid of base B and height AO or H, by Prop. IV. and its Cors. Therefore the whole pressure upwards is n times this column. Therefore, if the weight W be n times this column, the pressures downwards and upwards will balance each other, and there will be an equilibrium.

Prop. VI. If a body floats in a fluid it displaces as much of the fluid as is equal in weight to the body; and it presses downwards and is pressed upwards with a force equal to the weight of fluid displaced.

First, if the fluid be entirely under the body.

Let LM be a particle of the surface of the body; and on LM let a vertical column be erected, meeting the upper surface of the fluid in DE. Draw the horizontal section Ll of the column; and take KL perpendicular to LM to represent the pressure on LM, and draw KH perpendicular on the vertical line DL.



The force KL may be resolved into KH, HL, of which HL represents the vertical force; and the whole force on LM is to the vertical force on LM as KL to HL; that is, by Lemma 7, as LM to Ll; or as $DL \times LM$ to $DL \times Ll$. But the whole force on LM is equal to a column of fluid $DL \times LM$, by Cor. 2. to Prop. IV.; therefore the vertical force

on LM is equal to a column of fluid $DL \times Ll$; that is, to the column $EDLl$, by Lemma 6; that is, to the column $EDLM$, because the single particle LlM may be neglected, by Axiom 7.

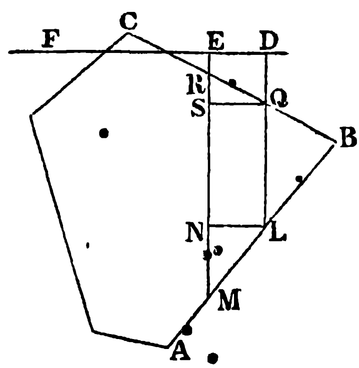
And, in like manner, the vertical pressure upon any other particle of the surface of the floating body is the weight of fluid equal to the vertical column which stands upon that particle, reaching up to the surface of the fluid.

And the whole vertical pressure upwards is equal to the sum of all these columns, that is, to the weight of the fluid displaced.

Secondly, if the fluid be above any part of the body, the excess of the vertical pressures upwards above the vertical pressures downwards is equal to the weight of the fluid displaced.

Let ABC be a vertical section of the body, EF the upper surface of the fluid, LM any particle of one of the lower surfaces of the body.

Draw the column $LDME$ vertical, meeting the upper surface of the fluid in DE, and cutting off a particle QR in the upper surface of the body. It may be proved, as in the former part, that the vertical pressure upwards on the particle LM is equal to the weight of the column of fluid $LDEM$. And in the same



manner it may be proved that the vertical pressure downwards on the particle QR is equal to the weight of the column of fluid $QDER$. Therefore the excess of the pressures upwards above the pressures downwards on this vertical column is the

excess of the weight of the column of fluid *LDEM* over that of *QDER*; that is, it is the weight of the column *LQRM*.

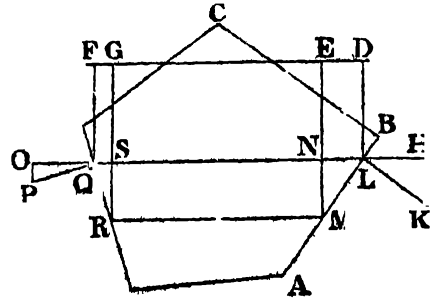
In the same manner, in any other vertical column, the excess of the pressure upwards above the pressure downwards is the weight of a quantity of fluid equal to the vertical column intercepted within the body. And the whole excess of the vertical pressures upwards is the sum of all such intercepted columns; that is, it is the weight of the fluid displaced by the body.

Therefore in all cases, the weight which can be supported by the pressure upwards, or by the excess of the pressure upwards, is the weight of the fluid displaced. But if a body float the weight of the body must be supported. Therefore the weight of the fluid displaced must be equal to the weight of the body.

And in this case the body presses downwards with its weight, that is, with the weight of the fluid displaced; and it is supported by an equal pressure upwards. Q.E.D.

PROP. A. If any horizontal prism be wholly or partially immersed in a fluid of uniform density, the horizontal pressures of the fluid on the sides of the prism destroy each other.

Let *ABC* be a vertical section of the prism perpendicular to its length, *EF* the upper surface of the fluid; *LM* any particle of one of the surfaces of the prism. Draw *LQ*, *MR* horizontal, cutting off *QR*, a particle of the opposite surface



of the prism. Draw LD, ME, QF, RG, vertical, to the upper surface of the fluid.

Since the plane ABC is perpendicular to the length of the prism, the pressures on the sides of the prism, which, by Axiom 5, are perpendicular to the sides, are in the plane ABC. Take KL, perpendicular and equal to LM, to represent the pressure on LM, and draw NLH horizontal, and KH vertical.

By Prop. IV. Cor. 2, the pressures on the particles LM, QR are as $LM \times LD$ and $QR \times QF$; that is, as LM and QR, because LD and QF are equal. Therefore, if a line KL, equal to LM, represent the force on LM, a line equal to QR will represent the force on QR. Let, therefore, PQ, perpendicular and equal to QR, represent the force on QR, and draw SQO horizontal and PO vertical.

Since MLK, LHK are right angles, the angles MLN, LKH are equal: and hence, LK being equal to LM, the triangles KHL, LNM are equal in all respects, so that $LH = MN$; also in like manner the triangles POQ, QSR are equal in all respects, so that $OQ = RS$. But MN is $= RS$; therefore $LH = OQ$.

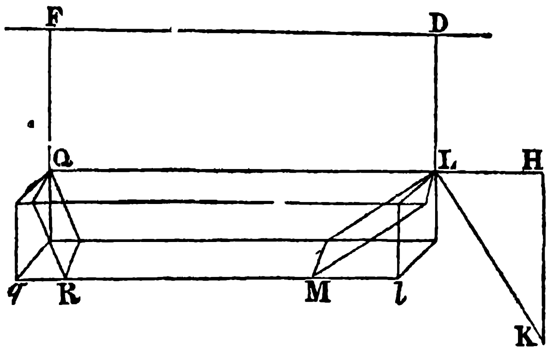
The force KL may be resolved into KH, HL, of which HL is the horizontal part; and the force PQ may be resolved into PO, OQ: of which OQ is the horizontal part; and OQ, HL have been shewn to be equal: therefore the horizontal forces on the two particles LM, QR are equal and opposite; therefore they destroy each other.

In the same manner, if any other lines be drawn horizontally in the plane of the figure, they will cut off, in the surface of the prism, opposite particles, on which the horizontal forces will destroy each other; and the horizontal forces on all such particles are the whole

horizontal pressures of the fluid on the sides of the prism. Therefore the whole horizontal pressures destroy each other. Q. E. D.

PROP. B. If a body bounded by plane surfaces be wholly or partially immersed in a fluid, the horizontal pressures of the fluid on the sides of the body, in any direction and its opposite, destroy each other.

Let LM be a particle of the immersed surface of the body, and on LM let a horizontal prism be constituted, (of which QL is one of the edges,) meeting



the opposite surface of the body, and cutting off the particle QR. Draw LD, QF, vertical lines, to the upper surface of the fluid. Take KL to represent the pressure on LM, and draw KH perpendicular on QL produced. And let Ll, Qq be the sections of the horizontal column by vertical planes.

The force KL may be resolved into KH, HL, of which HL is the horizontal force parallel to the line LQ. And the whole force on LM is to this horizontal force as KL to HL; that is, by Lemma 7, as LM to Ll, or as $LD \times LM$ is to $LD \times Ll$. But the whole pressure on LM is the weight of the column of fluid $LD \times LM$, by Prop. IV. Cor. 2. Therefore the horizontal force on LM parallel to LQ is the column $LD \times Ll$.

In like manner, it may be shewn that the horizontal force on QR, parallel to QL, is the weight of the column of fluid $QF \times Qq$, which is equal to the column $LD \times Ll$, because, by Lemma 5, Ll, Qq are equal.

Therefore the horizontal pressures on LM and QR, parallel to the line LQ, are equal and opposite, and therefore they destroy each other.

And, in the same manner, the horizontal pressures on any other two opposite particles, parallel to the line LQ, destroy each other. And the sum of all such horizontal pressures on opposite particles is the whole pressure on the surface of the body parallel to LQ. Therefore the whole of the horizontal pressures parallel to LQ destroy each other.

And, in like manner, the whole of the horizontal pressures parallel to any other horizontal line destroy each other.

Therefore the whole of the horizontal pressures in any direction and its opposite destroy each other. Q.E.D.

SCHOLIUM.

The last two Propositions are true of bodies bounded by curvilinear, as well as by plane surfaces. For the curvilinear figure is the limit of a polyhedral figure of a great number of sides. And what is true up to the limit is true of the limit.

Prop. C. When a body floats in a fluid, the centers of gravity of the body and of the fluid displaced are in the same vertical line.

When a body floats, its weight is balanced by the vertical pressures of the fluid on each particle of the immersed surface of the body; and these latter pres-

sures, by Prop. VI., are equal to the weight of vertical columns which would make up the fluid displaced. And the weights of these vertical columns will produce the same effect as if they were collected at their center of gravity, and acted upwards there, (Book I. Prop. E.), that is, at the center of gravity of the fluid displaced. And the weight of the body produces the same effect as if it were collected at its center of gravity, and acted downwards there. Therefore the two equal forces, one acting vertically upwards at the center of gravity of the fluid displaced, and the other acting vertically downwards at the center of gravity of the body, balance each other. But this cannot be, except they act in the same vertical line; therefore the two centers of gravity are in the same vertical line. Q.E.D.

SECTION II.

SPECIFIC GRAVITIES.

DEF. 1. The *Specific Gravity* of a substance is the proportion of the weight of any magnitude of that substance to the weight of the same magnitude of a certain standard substance (pure water).

For example, if a cubic foot of stone be three times as heavy as a cubic foot of pure water, the specific gravity of the stone is 3.

DEF. 2. The *density* is as the quantity of matter in a given magnitude, (D. I. Art. 13), and the quantity of matter is conceived to be as the weight: therefore the density of a body is as the specific gravity.

DEF. 3. When a body lighter than water is entirely immersed in water, it tends to ascend by a certain force which is called its *levity*.

PROP. VII. If M be the magnitude of a body, S its specific gravity, and W its weight, W varies as MS.

If the specific gravity increase in any ratio, the weight of a given magnitude increases in the same ratio, by the Definition of specific gravity; that is, the weight W varies as the specific gravity S; also if the specific gravity be given, the weight W increases as the magnitude M; therefore by the Introduction, Art. 57, if neither S nor M be given, W varies as MS.

COR. If A be the weight of a unit of magnitude of the standard substance (pure water), $W = AMS$.

For W is equal to MS with some multiplier, whole or fractional, by the Proposition. And when M is 1, and S is 1, by supposition $W=A$; therefore $W=AMS$ in all cases.

SCHOLIUM.

The weight of a cubic foot of water (A) is 63 pounds avoirdupois nearly.

The following is a list of the specific gravity of various substances; the standard (1) being pure water:—

Gold	19.3
Mercury	13 6
Lead	11.3
Silver	10.5
Copper	8.9
Iron	7.3
Marble	2.7
Water	1.0
Oak	1.2
Fir	.50
Cork	.24
Air	.00125 or $\frac{1}{800}$.

EXAMPLES.

. To find the weight of a cubic inch of silver.

The formula $W=AMS$ being applied in this case, A is 63 pounds, M is 1 inch, or $\frac{1}{1728}$ foot, S is 10.5;

$$\text{whence } W=\frac{63\times 10.5}{1728}\text{ pounds}=\frac{16\times 661.5}{1728}\text{ ounces}$$

$$=6.1 \text{ ounces.}$$

2. To find the weight of 10 feet square of gold leaf one thousandth of an inch thick.

$$M = 100 \times \frac{1}{12000}, \quad S = 19.3, \quad W = \frac{63 \times 19.3}{120} = \frac{1215.9}{120}$$
$$= 10.1 \text{ pounds.}$$

3. To find the weight of a cubical block of marble 1000 feet in the side.

$$W = 63 \times 1000^3 \times 2.7 = 130100000000 \text{ pounds}$$
$$= 58531250 \text{ tons.}$$

4. To find the weight of a column of air one inch base and 5 miles high.

$$W = 63 \times \frac{5 \times 5280}{144} \times .00125 = \frac{63 \times 5 \times 110}{3 \times 800}$$
$$= 14 \text{ pounds.}$$

PROP. VIII. When a body of uniform density floats on a fluid, the part immersed is to the whole body as the specific gravity of the body is to the specific gravity of the fluid.

For the magnitude of the part immersed is to that of the whole body as the fluid equal in bulk to the part immersed is to the fluid equal in bulk to the whole body. But the fluid equal in bulk to the part immersed is equal in weight to the whole body, by Prop. VI. Therefore the part immersed is to the whole as the weight of the body is to the weight of an equal bulk of fluid; that is, by the Definition of specific gravity, as the specific gravity of the body to that of the fluid. Q.E.D.

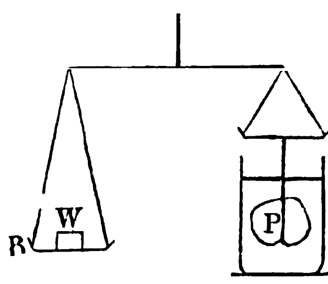
PROP. IX. When a body is immersed in a fluid, the weight lost in the fluid is to the whole weight of the body as the specific gravity of the fluid is to the specific gravity of the body.

When the body is wholly immersed, the pressure of the fluid vertically upwards is equal to the weight of a bulk of fluid equal to the body, by Prop. VI. But this pressure upwards diminishes the weight of the body when it is immersed in the fluid, and so much weight is lost. Therefore the weight lost in the fluid is equal to the weight of a bulk of fluid equal to the body. And the specific gravity of the fluid is to the specific gravity of the body, as the weight of a bulk of fluid equal to the body is to the weight of the body (Def.); that is, as the weight lost is to the whole weight. Q.E.D.

Prop. X. To describe the hydrostatic balance, and its use in finding the specific gravity of a body.

First, when the body is heavier than the fluid in which it is weighed (water).

The hydrostatic balance is a balance in which a body (P) can be weighed, either out of water, in the scale A, in the usual manner, or in the water (as at P).



In order to find the specific gravity of any body, let it be weighed out of water, and in water; the difference is the weight lost in water; and hence the specific gravity is known by the last Proposition.

Cor. If U be the weight of the body out of water, V the weight in water, W the weight of an equal bulk of water, and S the specific gravity,

$$W = U - V, \text{ and } S = \frac{U}{W} = \frac{U}{U - V}.$$

Secondly, when the body is lighter than water.

Let the proposed body be weighed out of water; also let it be fastened to a *sinker* of which the weight in water is known, and let the compound body be weighed in water.

The excess of the weight in water of the sinker, above the weight in water of the compound body, is the levity of the proposed body: for by attaching the proposed body, its levity or tendency upwards in water diminishes the weight in water of the sinker.

The levity of the proposed body, together with its weight out of water, are equal to the weight of an equal bulk of fluid; for the levity of the body in water is the excess of the pressure upwards above the pressure downwards; that is, the excess of weight of an equal bulk of fluid above the weight (out of water) of the body.

Hence the weight of an equal bulk of water is known, and hence the specific gravity, by the Definition of specific gravity.

COR. If U be the weight of the body out of water, Q the weight of the sinker in water, and R the weight of the compound body in water. The levity of the proposed body is $Q - R$. Hence $Q - R + U$ is the weight of an equal bulk of fluid; and

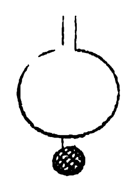
$$S = \frac{U}{Q - R + U}.$$

EX. The weight of the body is 2, and by attaching to it a sinker which weighs 4 in water, the compound body weighs 3 in water. Therefore the levity of the body is 4 − 3 or 1, and the weight of an equal bulk of fluid is 2 + 1 or 3. Hence the specific gravity is $\frac{2}{3}$.

PROP. XI. To describe the common hydrometer, and to shew how to compare the specific gravities of two fluids by means of it.

The common *Hydrometer* is an instrument consisting of a body and a slender stem, and of such specific gravity that in the fluids for which it is to be used, it floats with the body wholly immersed and the stem partially immersed.

The part immersed is to the whole as the specific gravity of the body is to the specific gravity of the fluid (Prop. VIII); and if the specific gravity of the fluid vary, the part immersed will vary in the inverse ratio of the specific gravity.



But since the stem is slender, small variations of the part immersed will occupy a considerable space in the stem, and will be very easily ascertained.

If the magnitude of the whole instrument be represented by 4000 parts and each of the divisions of the stem by 1 such part; and if the whole length of the stem contain 100 such parts, the instrument will measure with great accuracy specific gravities of fluids within certain limits.

Let the fluids be compared with a certain "proof" standard, as 50, in the middle of the scale. If the instrument sink to 30, the specific gravity of the fluid is known. For the part immersed is 4000 – 30, or 3970: and the "proof" fluid, the part immersed is 4000 – 50, or 3950. Therefore the specific gravity of the fluid is to that of "proof fluid" as 3950 to 3970, or as 395 to 397.

SECTION III.

ELASTIC FLUIDS.

PROP. D. *Inductive Principle I.* Water and other liquids have weight in all situations.

The facts included in this induction are such as the following:—

(1). Water falls in air as solid bodies do.

(2). A bucket of water held in air is heavy and requires to be supported in the same manner as a solid body.

(3) A bucket of water held in water appears less heavy than in air, and may be immersed so far as not to appear heavy at all.

(4). A lighter liquid remains at rest above a heavier, as oil of turpentine upon water.

(5). The bodies of divers plants, and other organised bodies, though soft, are not compressed or injured under a considerable depth of water.

The different effects (2) and (3) led to the doctrine that all the elements have their *proper places*, the place of earth and heavy solids being lowest, of heavy fluids next above, of light fluids next, of air next; and that the elements do not gravitate when they are in their proper places, as water in water; but that water in air, being out of its proper place, gravitates, or is heavy. In this way also (1) and (4) were explained.

But it was found that this explanation was not capable of being made satisfactory; for—(6) a solid body of the same size and weight as the bucket of water

in (3) gave rise to the same results; and these could not be explained by saying that the solid body was in its proper place.

These facts can be distinctly explained and rigorously deduced, by introducing the *Idea* of *Fluid Pressure;* and the *Principle* that water is a heavy fluid, its weight producing effects according to the laws of fluid pressure.

For on this supposition (1) and (2) are explained, because water is heavy; and (3) is explained by the pressure of the fluid upwards against the bucket, according to Propositions I., II., IV.

Also it may be shewn by experiment that in such a case as (4) the lighter fluid increases the pressure which is inserted in the lower fluid.

Facts of the nature of (5) are explained by considering than an equal pressure is exerted on all parts of the organised structure, in opposite directions; such pressures balance each other, and no injury results to the structure, except in some cases a general contraction of dimensions. If there be a communication between the fluids which are within the structure and the fluid in which it is placed, these pressures are exerted from within as well as from without, and the balance is still more complete.

Also all the other observed facts were found to confirm the idea of fluids, considered as heavy bodies exerting fluid pressure: thus it was found—(7) that a fluid presses downwards on a lighter body which is entirely immersed; and presses upwards on a heavier body which is partially immersed; and presses in all directions against surfaces, according to the deductive Propositions which we have demonstrated to obtain in a heavy fluid.

PROP. XII. *INDUCTIVE PRINCIPLE II.* Air has weight.

The facts included in this induction are such as the following:—

(1). We, existing in air, are not sensible of any weight belonging to it.

(2). Bubbles of air rise in water till they come to the surface.

(3). If we open a cavity, as in a pair of bellows, the air rushes in.

(4). If in such a case air cannot enter and water can, the water is drawn in; as when we draw water into a tube by suction, or into a pump by raising the piston.

(5). If a cavity be opened and nothing be allowed to enter, a strong pressure is exerted to crush the sides of the cavity together.

If facts (1) and (2) were explained at first by saying that the *proper place* of air is above water; that when it is in its proper place, as in (1), it does not gravitate (as in Prop. D.), but that when it is below its proper place, as in (2), it tends upwards to its place; the facts (3) (4) (5) were explained by saying that *nature abhors a vacuum.*

But it was found by experiment:—

(6). That water could not by suction or by a pump be raised more than 34 feet; and stood at that height with a vacuum above it.

(7). That mercury was supported in a tube with

a vacuum above it, at the height of 30 inches (Torricelli's experiment).

(8). That at the top of a high hill this column of mercury was less than 30 inches (Pascal's experiment).

These facts overturned the explanation derived from nature's horror of a vacuum; for men could not suppose that nature abhorred a vacuum less at the top of a hill than at the bottom, or less over 34 feet of water than over one foot.

But all the facts were distinctly explained and rigorously deduced by adopting the *Idea* of fluid pressure, and the *Principle* that air has weight, its weight producing its effects according to the laws of fluid pressure. This will be seen in the Deductive Propositions which we shall demonstrate as the consequences of assuming that air has weight.

The Inductive Proposition was further confirmed by—(9) experiments with the air-pump; for it appeared that as the receiver was exhausted the mercury in the Torricellian experiment fell.

PROP. XIII. *INDUCTIVE PRINCIPLE III.* Air is elastic; and the elastic force of air at a given temperature varies as the density.

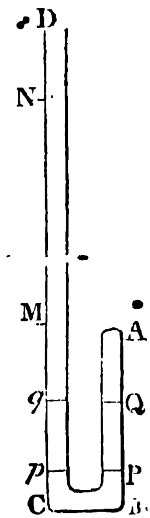
The facts which shew air to be elastic are such as follow:—

(1). A bladder containing air may be contracted by pressure, and expands again when the pressure is removed.

(2). A tube closed above and open below, and containing air, being immersed in water, the air contracts as the immersion is deeper, and expands again when the tube is brought to the surface.

(3). If a close vessel containing water and air, fitted with a tube making a communication between the water and the exterior, be placed in the exhausted receiver, the water is expelled through the tube.

The principle that the elastic force increases *in proportion to* the density, was experimentally proved (first by Boyle*) in the following manner:—



A uniform tube *ABCD* was taken, closed at *A* and open at *D*, and bent so that *BA* and *CD* were upright at the same time. Quicksilver was poured in, so that its ends stood at *M* and *P*. Again, more quicksilver was poured in, so that its ends stood at *N* and *Q*. And *Pp*, *Qq* being horizontal, *AQ* and *Nq* were measured.

And the observations of the results of this experiment were registered as in the two first columns of the annexed table,

(1) *A Q* in.	(2) *N q* in.	(3) Barom. in.	(4) Press. in.
12	0	30	30
10	6	30	36
8	15	30	45
6	30	30	60
4	60	30	90

The whole pressure on the air *AQ* at *Q* is the pressure of the column of mercury *Nq*, together with the pressure of the atmosphere upon *N*. The latter pressure being taken to be equal to 30 inches of mer-

* Shaw's Boyle, Vol. II. p. 671.

cury, as in column (3), and added to Nq, we have, as in column (4), the pressure upon the air AQ. And it appears by comparing columns (1) and (4) that this pressure is always inversely as the space occupied by the air;

$$\text{for } 10 : 6 :: 60 : 36,$$

and so for any other of the observations.

Now the pressure on the air AQ at Q is balanced by the elastic force of the air in AQ; these two forces acting upon the same surface, namely the surface of the mercury at Q. And the pressure upon Q has been found to be inversely as AQ; therefore the elasticity of the air in AQ is inversely as AQ; that is, inversely as the space occupied.

The quantity of air remaining the same, the density is inversely as the space occupied: therefore the elastic force is as the density.

PROP. XIV. *INDUCTIVE PRINCIPLE IV.* The elastic force of air is increased by an increase of temperature.

The facts included in this induction are such as the following:—

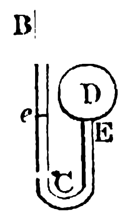
(1). If a bladder partly full of air be warmed it becomes more completely full.

(2). If an inverted vessel confining air in water be warmed the air escapes in bubbles.

It was experimentally ascertained *how much* the elastic force of air is increased by heat (first by Amontons*) in the following manner:—

* Mem. de l'Acad. Roy. des Sciences de Paris. 1699. p 113.

A bent tube *ABC*, with a bulb *D* containing common air, was filled with mercury from *B* to *E*, *B* being 3 inches higher than the horizontal plane *Ec*. The bulb was then placed in boiling water, and it was found that a small portion of the mercury was driven out of the bulb, so that the extremity of the column was elevated to *F*, *BF* being 11 inches.



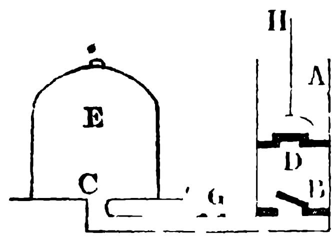
The air occupied very nearly the same space in the last case as in the first; for the bore of the tube was very small, and the surface of the mercury continued nearly in the same position at *E*. The pressure on the air in *D* at first is the pressure of the atmosphere (30 inches of mercury,) together with the weight of the column *Be* (3 inches;) therefore it is 33 inches. And the pressure on the air in *D* when immersed in boiling water is, in the same manner 30 inches, together with the weight of the column *Fe* (which is 14 inches;) that is, it is 44 inches; that is, air, in this experiment, has its elasticity increased from 33 to 44, by heating the water to boiling: that is, the elasticity was increased one third.

PROP. XV. To describe the construction of the air-pump and its operation.

A *valve* is an appendage to an orifice, closing it, and opening in such a manner as to allow fluid to pass through the orifice in one direction and not in the opposite direction.

A *piston* is a plug capable of sliding in an orifice or tube so as to produce or remove fluid pressure.

The *Air-pump* consists of a barrel and piston with valves, the *suction-pipe* communicating with a close vessel called the *receiver*.



Let *AB* be the barrel, *B* the inwards-opening valve at the bottom of the barrel, *D* the piston with its outwards-opening valve, *BC* the pipe, *E* the receiver.

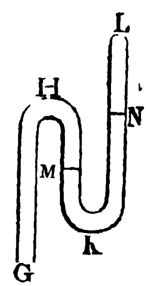
The piston *D* being in its lowest position, is raised to its highest position by the handle *H*. During the rise no air is admitted at *D*; and the air in *CD*, by its elasticity, expands and follows the piston in its ascent, passing through the valve *B*; and thus air is drawn out of the receiver *E*

The piston is then made to descend again to its lowest position: no air returns through the valve *B*, and the air in *BD* escapes by the valve at *D*.

The piston is again raised, and more air is drawn out of *E* as before: and so on without limit.

PROP. E. To explain the construction of the siphon-guage.

The *Siphon-gauge* is a twice-bent tube, closed at one end, and containing fluid, fixed to an air-pump or other machine, to determine the degree of rarefaction of the air.

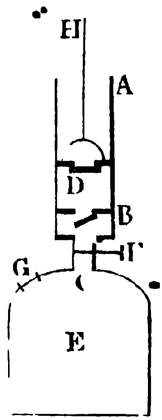


Let *GHKL*, closed at *L*, be the siphon-gauge, (fixed to *G* in the last or in the next Proposition), and let *MKN* be a portion of the tube filled with mercury, *LN* being a vacuum. Then the vertical height of *N* above *M* measures the density of the air in *GHM*.

If *GHM* were a vacuum (that is, if the exhaustion in the air-pump were complete) *M*, *N* would be at the same level.

Prop. XVI. To describe the condenser and its operation.

The *Condenser* consists of a barrel and piston with valves, opening the contrary way from those of the air-pump, and communicating by a pipe with a closed receiver.



Let *AB* be the barrel, *B* the outwards-opening valve at the bottom of the barrel, *D* the piston with its inwards-opening valve, *BC* the pipe, *E* the receiver.

The piston *D* being in its highest position, is forced to its lowest position by the handle *H*. During the descent no air escapes through *D*, and the air in *BD* is driven through the valve *B*, and increases the quantity in the receiver *E*.

The piston is then made to ascend, and no air enters the barrel at *B*, because the valve opens outwards; but air enters the barrel *BD* by the valve *D*.

The piston is again forced down, and more air is driven into the receiver *E* as before: and so on without limit.

The pipe *BC* has a stop-cock *F*, and when this is closed, the pump may be screwed off, after the condensation is made.

Prop. XVII. To explain the construction of the common barometer, and to shew that the

mercury in the tube is sustained by the pressure of the air on the surface of the mercury in the basin.

A *Barometer* is a (glass) tube, closed at one end and open at the other, which, being filled with a fluid (as mercury) is inverted with its open end in a basin. In any place, the fluid stands at a certain height (if the tube be long enough,) leaving a vacuum above.

Since the air has weight, it presses upon the surface *CD* of the mercury in the basin, and this pressure is resisted by the pressure of the column of mercury *PM*, arising from its weight. The mercury in the tube is sustained by the pressure of the mercury in the basin *CD*, which pressure again is sustained by the pressure of the atmosphere on the surface of the mercury in the basin.

PROP. XVIII. In the common barometer, the pressure of the atmosphere is measured by the height of the column of mercury above the surface of the mercury in the basin.

Let *AM* be the vertical tube, *A* its closed end, *CD* the basin and *MP* the height at which the fluid stands.

The upper parts of the atmosphere are less dense than the lower; but so long as the whole is in equilibrium, this condition does not affect the laws of fluid pressure; and Propositions I., II., III., IV., V., VI., of this Book will be still true.

Take, on the surface of the basin, *NO* equal to *NM*, the horizontal section of the tube; and suppose

a tube HN, with vertical sides, standing on the base NO, to be continued upwards to the limits of the atmosphere. By Axiom 3, if all the rest of the atmosphere become rigid the pressure is not altered; and hence by Prop. IV., the pressure upon NO is equal to the weight of the column HN. But on this supposition, the pressures on MN, NO are equal, by Prop. I. And the pressure on MN is equal to the weight of the vertical column of mercury MP. Therefore the weight of the column of mercury is equal to the weight of the column of atmosphere on the same base. Therefore the weight of the column of atmosphere is measured by the weight of the column of mercury; that is, the pressure of the atmosphere on a surface equal to the section of the tube made at the surface of the mercury in the basin, is equal to the weight of the vertical column of mercury which stands on the same section.

Therefore the pressure of the atmosphere is measured by the weight of the column of mercury, that is, by the height, if the section and the density continue constant; for the weight of a column is as section × height × density.

Cor. 1. If, instead of mercury, the tube be filled with any other fluid, as water, the fluid will stand at such a height as to support the weight of the atmosphere; and the height will be greater as the density of the fluid is less.

The mean height of the mercury-barometer being 30 inches, and the specific gravity of mercury 13.6, the mean height of the water-barometer is 13.6×30 inches $= 408$ inches $= 34$ feet.

Cor. 2. If the tube AM be not vertical, the Proposition is still true, the vertical height of A above

M being still taken for the height of the fluid; for the pressure on *MN* is the same as if *AM* were vertical, by Prop. II.

Cor. 3. If the portion of the tube *AP**, instead of being empty, contain air of less density than the atmosphere, a column of fluid *PM* will still be sustained, smaller than the column where *AP* is a vacuum; for if *P* were to descend to *M*, the pressure on *MN* would be less than the pressure on *NO*, which is impossible.

Cor. 4. If a tube *AM*†, inclined or vertical, having its lower end, *M*, immersed in a fluid, and its upper end, *A*, closed, be full of water; the water will be supported if the vertical height of *A* above *M* be less than the height of the water-barometer.

In this case if *HM* be the height of the water-barometer, and if *AP*, drawn horizontal, meet *HM* in *P*, the pressure upwards on the fluid at *M* would support a column of water of the vertical height *HM*. The pressure arising from the water in *AM* is equivalent to the weight of a column of the height *PM* (Prop. IV). Therefore the pressure upwards at *M* will support the column *AM*; and will, besides, produce at *A*, a pressure upwards equal to the weight of a column *HP*.

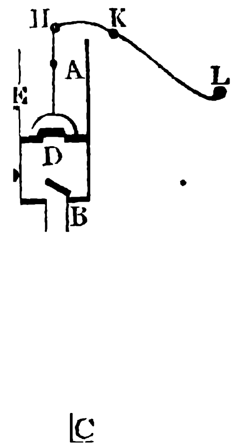
Cor. 5. If the interior of a tube *AM*†, inclined or vertical, having its lower end *M* immersed in water, and its upper end *A* closed, become a *vacuum*, (that is, empty of air as well as other fluids) the fluid will rise in the tube; and will fill the whole tube or part of it, according as the vertical height of the closed end is less or greater than the height of the water-barometer.

* See fig. p. 114. † See fig. p. 119.

For when the surface within the tube is not pressed by the column of fluid requisite to produce equilibrium, the pressure on the other parts of the surface will prevail, and drive the fluid into the vacuum. (Axiom 3.)

Prop. XIX. To describe the construction of the common pump, and its operation.

The *Common Pump* consists of a cylindrical barrel *AB*, closed at bottom with an upwards-opening valve *B*, and of a piston *D* with an upwards-opening valve, which moves up and down in the barrel. A *suction-pipe BC* passes downwards from the valve *B* to the *well* at *C*, and the water which rises above the piston is delivered by the *spout E*.



The operation of the pump is as follows. The piston *D* being in its lowest position, is raised to its highest position by means of the lever *HKL*. Since the valve *D* opens upwards, no air is admitted at *D* during this rise; and since the valve *B* opens upwards, the air which occupied *CD* follows the piston in its ascent; in consequence of its elasticity (Prop. XIII) it expands, and its pressure on the water at *C* is diminished. Hence the water in the suction-pipe rises by the pressure of the atmosphere on the surface of the well to some point *F*. (Cor. 3 to Prop. XVIII).

The piston is then made to descend to its lowest position, the valve *B* is closed, and therefore the quantity of air in *FB* is not changed, and the water remains at *F*, while the air in *BD* escapes by the valve at *D*.

The piston is then again raised, the air in *DF*

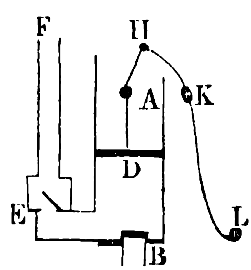
expands as before, and the surface of the water at F comes to a new position at G.

The same movements being repeated, the water will again rise; and so on, till it reaches the piston D, after which time the piston in its ascent will lift the water, and when it has lifted it high enough, will deliver it out at the spout E.

COR. The water in the common pump is raised by the weight of the atmosphere, and cannot be raised to a height greater than that of the water-barometer. (See Prop. XII.) The height of the water-barometer is 34 feet (Prop. XVIII. Cor. 1.)

PROP. XX. To describe the construction of the forcing pump and its operation.

The *Forcing Pump* consists of a cylindrical *barrel* AB, closed at bottom with an upwards-opening valve B; of a *piston* D with no valve; and of a *spout* E with an outwards-opening valve. The piston moves up and down, and the suction-pipe descends from the bottom of the barrel to the well, as before, and the spout carries the water upwards.



The operation of the pump is as follows. The piston D, in ascending from its lowest to its highest position, draws the water after it as in the common pump. When the piston descends, the air is forced out at the valve E; and after a certain number of ascents, the water comes into the barrel AB. When the piston next descends it forces the water through the valve E, and continues afterwards to draw the water through the valve B in its

rise, and to extrude it through the valve E in its descent, by which means it is forced into the tube EF, which may be upright, or in any other position.

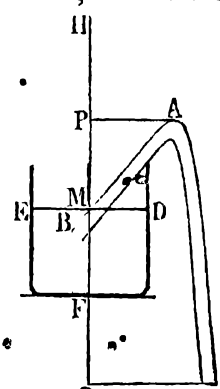
COR. By means of the forcing pump water may be raised to any height; the tube EF being prolonged upwards, and an adequate force applied to force the piston downwards.

PROP. XXI. To describe the siphon and its action.

A *Siphon* is a bent tube, open at both ends, and capable of being placed with one end in a vessel of fluid, and the other end lower than the upper surface of the fluid in the vessel.

Let BAC be the bent tube placed so that the end B is immersed in the water FED, and the outer end C is below the surface ED.

If the tube BAC be filled with water, and if the vertical height of the portion MA be less than the height of the water-barometer, the tube will act as a siphon, that is, the water will constantly run through the tube BAC and out at C.



For the tube being filled with water, let the end C be stopped; and let HM be the height of the water-barometer; AP, CQ horizontal. Suppose HM to be a column of water of the height of the water-barometer; and suppose the water in the tube $HMAC$ to remain fluid, while all the rest becomes rigid: the pressures at M, A, and C will not be altered by this supposition (Axiom 3). But on this supposition the pressure downwards at C is equal to the height of a vertical column HQ

(Prop. II.) And the pressure of the atmosphere upwards at C is equal to the weight of a vertical column HM (Prop. XVIII. Cor. 2). Therefore the column AC, being acted upon by a pressure downwards equal to the weight of a column HQ, and a pressure upwards equal to the weight of a column HM, if the tube be opened at C, the former will prevail and the column AC will descend.

Also the column MA will ascend, so that no interval shall exist in the fluid at A. For the interval, if any should take place, must be a vacuum, since the air has no access to it. And since the vertical height of MA is less than that of the water-barometer, by Cor. 5 to Prop. XVIII., the fluid will rise in the tube and will fill this vacuum.

Therefore the whole fluid MAC will move along the tube and flow out at C.

Cor. 1. The fluid in the siphon BAC will be urged in the direction BAC by a force equal to a column of fluid MQ.

Cor. 2. If the vertical height of MA be greater than that of the water-barometer, there will be a vacuum formed above the fluid at A (Prop. XVIII. Cor. 5), and the siphon will not act.

Cor. 3. Also, if instead of water and the water-barometer, we had taken any other fluid and the corresponding barometer, the reasoning, and the result, would have been the same as above.

Prop. F. *Inductive Principle V.* Many (or all) fluids expand by heat; and the amount of expansion at the heat at which water boils, and at the heat which ice melts, are each a fixed quantity.

The former part of this proposition is proved by including the fluids in bulbs, which open into a slender tube; for a small expansion of the fluid in the bulb is easily seen, when it takes place in the slender tube.

It was at first supposed, that when a fluid is exposed to heat, (as, for instance, when a vessel of water is placed on the fire,) a constant addition of heat takes place, increasing with the time during which the fire operates.

But it appeared, that when a tube containing air is placed in water thus exposed to heat, the expansion of the air (observed in the way described in Prop. XIII) goes on till the fluid boils, after which no additional expansion takes place.

This fact is explained by assuming the expansion of air as the *Measure* of heat, and by adopting the *Principle* that the heat of boiling water is a fixed quantity.

This principle was first experimentally established by Amontons. Afterwards it was ascertained by Fahrenheit (1711), and others, that the expansion of oil, spirit of wine, mercury, at the heat at which water boils, is a fixed quantity; and hence Fahrenheit made the *boiling point* of water one of the fixed points of his thermometers, which were filled with spirit of wine or with mercury.

For another fixed point he took the cold produced by a mixture of ice, water and salt; and he assumed this to be the *point of absolute cold.*

But it was found by Reaumur (1730), that the *freezing point* of water, or the melting point of ice, is more fixed than the point of absolute cold determined

in the above manner. This was proved in the same manner in which the heat of boiling water had been proved to be a fixed point. The *freezing point* was then adopted as one of the fixed points of the measure of heat.

PROP. XXII. To shew how to graduate a common thermometer.

The common *Thermometer* is an instrument consisting of a bulb and a slender tube of uniform thickness, containing a fluid (as mercury or spirits of wine) which expands by heat and contracts by cold, so that its surface is always in the tube*.

Let the instrument be placed in boiling water, and let the point to which the surface of the fluid expands in the tube be marked as the *boiling point*.

Let the instrument be immersed in melting ice, and let the point to which the surface of the fluid contracts in the tube be marked as the *freezing point*.

For *Fahrenheit's division*, divide the interval between the freezing point and the boiling point into 180 equal parts; and continue the scale of equal parts upwards and downwards. Place 0 at 32 parts below the freezing point, 32 at the freezing point, 212 at the boiling point; and the other numbers of the series at other convenient points, and the scale is graduated, the numbers expressing degrees of heat according to the place of the surface of the fluid in the tube.

For the *centigrade division*, divide the interval between the freezing and boiling points into 100 equal parts; mark the freezing point as 0 degrees, the boiling point as 100 degrees, and so on as before.

* In practice, the part of the thermometer not occupied by the thermometrical fluid is rendered a vacuum.

PROP. XXIII. To reduce the indications of Fahrenheit's thermometer to the centigrade scale, and the converse.

To reduce Fahrenheit to centigrade, subtract 32, which gives the number of degrees above the freezing point: and multiply by $\frac{5}{9}$, because 180 degrees of Fahrenheit are equal to 100 centigrade.

Thus

$$59^0 F = 27^0 F \text{ above } 32^0 F = \frac{5 \times 27^0}{9} \text{ centig. above } 0$$
$$= 15^0 \text{ cent.}$$

To reduce centigrade to Fahrenheit, multiply by $\frac{9}{5}$, which gives the number of Fahrenheit's degrees above the freezing point, and add 32, which gives the number above Fahrenheit's zero.

Thus

$$60 \text{ centig.} = 90F \text{ above freezing} = 90F + 32F = 122F.$$

NOTES RESPECTING THE EXAMINATIONS IN THE PRECEDING SUBJECTS.

UNIVERSITY REGULATIONS.

PLAN OF EXAMINATION FOR QUESTIONISTS WHO ARE NOT CANDIDATES FOR HONORS.

1. That the subjects of the Examination shall be the first fourteen, or the last fourteen Chapters of the Acts of the Apostles, and one of the longer, or two or more of the shorter Epistles of the New Testament, in the original Greek, one of the Greek and one of the Latin Classics, three of the six Books of Paley's Moral Philosophy, the History of the Christian Church from its Origin to the assembling of the Council of Nice, the History of the English Reformation, and such mathematical Subjects as are prescribed by the Grace of April 19, 1837, at present in force.

2. That in regard to these Subjects, the appointment of the Division of the Acts—of the Epistle or Epistles—of the Books of Paley's Moral Philosophy, and both of the Classical Authors and of the portions of their Works, which it may be expedient to select, shall be with the persons who appoint the Classical Subjects for the Previous Examination.

3. That public notice of the Subjects so selected for any year shall be issued in the last week of the Lent Term of the year next but one preceding.

4. That the Examination shall commence on the Wednesday preceding the first Monday in the Lent Term.

5. That on the Monday previous to the commencement of the Examination the Examiners shall publish the names of the persons to be examined, arranged in alphabetical order, and separated into two divisions.

6. That the distribution of the Subjects and Times of Examination shall be according to the following Table:

	Div.	9 to 12.	Div.	12½ to 3½.
Wednesday.	1	Euclid	2	Greek Subject.
Thursday...	1	Greek Subject	2	Euclid
Friday......	1	Mechanics and Hydrostatics	2	Latin Subject
Saturday ...	1	Latin Subject	2	Mechanics and Hydrostatics
Monday.....	1	Paley *and Eccles. History*	2	Acts *and Epistle or Epistles.*
Tuesday	1	Acts *and Epistle or Epistles.*	2	Paley *and Eccles. History...*
Wednesday.	1	Arithmetic and Algebra	2	Arithmetic and Algebra

7. That the Examination shall be conducted entirely by printed papers.

8. That the Papers in the Classical Subjects and in the Acts and Epistles shall consist of passages to be translated, accompanied with such plain questions in Grammar, History, and Geography, as arise immediately out of those passages.

9. That the Papers in the Mathematical subjects shall consist of questions in Arithmetic and Algebra, and of Propositions in Euclid, Mechanics, and Hydrostatics, according to the annexed schedule.

10. That no person shall be approved by the Examiners, unless he shew a competent knowledge of all the subjects of the Examination.

11. That there shall be three additional Examinations in every year; the first commencing on the Thursday preceding Ash-Wednesday, the second on the Thursday preceding the Division of the Easter Term, and the third on the Thursday preceding the Division of the Michaelmas Term.

12. That in these additional Examinations the distribution of the subjects and the hours of the Examination shall be at the discretion of the Examiners, the subjects being the same as at the Examination in the preceding January.

13. That no person shall be allowed to attend any Examination whose name is not sent by the Prælector of his College to the Examiners before the commencement of the Examination.

14. That in every year at the first Congregation after the 10th day of October, the Senate shall elect four Examiners, (who shall be Members of the Senate, and nominated by the several Colleges according to the cycle of Proctors and Taxors) to assist in conducting the Examinations of the three following terms.

15. That two of these Examiners shall confine themselves to the Classical Subjects, and two to Paley's Moral Philosophy, Ecclesiastical History, the Acts of the Apostles, and the Epistles.

16. That the two Examiners in the Mathematical Subjects, at the Examination in January, be as hitherto the Moderators of the year next but one preceding; and that at the other three Examinations the Moderators for the time being examine in the Mathematical Subjects.

17. That each of the six Examiners shall receive £20 from the University Chest.

18. That the Pro-Proctors and two at least of the Examiners attend in the Senate-House during each portion of the Examination in January.

19. That the first Examination, under the Regulations now proposed, [that is, in the theological subjects] shall take place in the Lent Term of 1846.

SCHEDULE OF MATHEMATICAL SUBJECTS of Examination, for the degree of B.A of Persons not Candidates for Honors.

ARITHMETIC.

Addition, subtraction, multiplication, division, reduction, rule of three; the same rules in vulgar and decimal fractions: practice, simple and compound interest, discount, extraction of square and cube roots, duodecimals: *together with the proofs of the rules and the reasons for the processes employed* *.

ALGEBRA.

1. Definitions and explanation of algebraical signs and terms.
2. Addition, subtraction, multiplication and division of simple algebraical quantities and simple algebraical fractions.
3. Algebraical definitions of ratio and proportion.
4. If $a : b :: c : d$ then $ad = bc$, and the converse:

$$\begin{aligned} &\text{also } b : a :: d : c,\\ &\text{and } a : c :: b : d\\ &\text{and } a+b : b :: c+d : d.\end{aligned}$$

5. $$\begin{aligned} &\text{If } a : b :: c : d,\\ &\text{and } c : d :: e : f,\\ &\text{then } a : b :: e : f.\end{aligned}$$

6. $$\begin{aligned} &\text{If } a : b :: c : d,\\ &\text{and } b : e :: d : f,\\ &\text{then } a : e :: c : f.\end{aligned}$$

* Added by Grace, March 20, 1846.

7. Geometrical definition of proportion. (Euc. Book v. Def. 5.)

8. If quantities be proportional according to the algebraical definition, they are proportional according to the geometrical definition.

9. Definition of a quantity *varying as* another, *directly*, or *inversely*, or as two others *jointly*.

10. *Easy equations of a degree not higher than the second involving one or two unknown quantities and questions producing such equations**.

EUCLID'S ELEMENTS.

Book I. II. III.

Book VI. Props. 1. 2. 3. 4. 5. 6.

MECHANICS.

Definition of Force, Weight, Quantity of Matter, Density, Measure of Force.

The Lever.

Definition of the Lever.

Axioms.

Prop. 1. A horizontal prism or cylinder of uniform density will produce the same effect by its weight as if it were collected at its middle point.

Prop. 2. If two weights acting perpendicularly on a straight lever on opposite sides of the fulcrum balance each other, they are inversely as their distances from the fulcrum; and the pressure on the fulcrum is equal to their sum.

Prop. 3. If two forces acting perpendicularly on a straight lever in opposite directions and on the same side of the fulcrum balance each other, they are inversely as their distances from the fulcrum; and the pressure on the fulcrum is equal to the difference of the forces.

Prop. 4. To explain the kind of levers.

Prop. 5. If two forces acting perpendicularly at the extremities of the arms of any lever balance each other, they are inversely as the arms.

Prop. 6. If two forces acting at any angles on the arms of any lever balance each other, they are inversely as the perpendiculars drawn from the fulcrum to the directions in which the forces act.

Prop. 7. If two weights balance each other on a straight lever when it is horizontal, they will balance each other in every position of the lever.

* Added by Grace, March 20, 1846.

Composition and Resolution of Forces.

Definition of Component and Resultant Forces.

Prop. 8. If the adjacent sides of a parallelogram represent the component forces in direction and magnitude, the diagonal will represent the resultant force in direction and magnitude.

Prop. 9. If three forces, represented in magnitude and direction by the sides of a triangle, act on a point, they will keep it at rest.

Mechanical Powers.

Definition of Wheel and Axle.

Prop. 10. There is an equilibrium upon the wheel and axle when the power is to the weight as the radius of the axle to the radius of the wheel.

Definition of Pulley.

Prop. 11. In the single moveable pulley where the strings are parallel, there is an equilibrium when the power is to the weight as 1 to 2.

Prop. 12. In a system in which the same string passes round any number of pulleys and the parts of it between the pulleys are parallel, there is an equilibrium when power (P) : weight (W) :: 1 : the number of strings at the lower block.

Prop. 13. In a system in which each pulley hangs by a separate string and the strings are parallel, there is an equilibrium when $P : W :: 1 :$ that power of 2 whose index is the number of moveable pulleys.

Prop. 14. The weight (W) being on an inclined plane and the force (P) acting parallel to the plane, there is an equilibrium when $P : W ::$ the height of the plane : its length.

Definition of Velocity.

Prop. 15. Assuming that the arcs which subtend equal angles at the centres of two circles are as the radii of the circles, to shew that if P and W balance each other on the wheel and axle, and the whole be put in motion, $P : W :: W$'s velocity : P's velocity.

Prop. 16. To shew that if P and W balance each other in the machines described in Propositions 11, 12, 13, and 14, and the whole be put in motion, $P : W :: W$'s velocity in the direction of gravity : P's velocity.

The Centre of Gravity.

Definition of Centre of Gravity.

Prop 17. If a body balance itself on a line in all positions, the centre of gravity is in that line.

Prop. 18 To find the centre of gravity of two heavy points; and to shew that the pressure at the centre of gravity is equal to the sum of the weights in all positions.

Prop 19. To find the centre of gravity of any number of heavy points; and to shew that the pressure at the centre of gravity is equal to the sum of the weights in all positions.

Prop 20. To find the centre of gravity of a straight line.

Prop. 21 To find the centre of gravity of a triangle.

Prop. 22 When a body is placed on a horizontal plane, it will stand or fall, according as the vertical line, drawn from its centre of gravity, falls within or without its base.

Prop. 23. When a body is suspended from a point, it will rest with its centre of gravity in the vertical line passing through the point of suspension.

Hydrostatics.

Definitions of Fluid; of elastic and non-elastic Fluids.

Prop. 1. Fluids press equally in all directions.

Prop. 2. The pressure upon any particle of a fluid of uniform density is proportional to its depth below the surface of the fluid.

Prop. 3. The surface of every fluid at rest is horizontal.

Prop. 4. If a vessel, the bottom of which is horizontal and the sides vertical, be filled with fluid, the pressure upon the bottom will be equal to the weight of the fluid.

Prop. 5. To explain the *hydrostatic paradox*.

Prop. 6. If a body floats on a fluid, it displaces as much of the fluid as is equal in weight to the weight of the body; and it presses downwards and is pressed upwards with a force equal to the weight of the fluid displaced.

Specific Gravities.

Definition of Specific Gravity.

Prop. 7 If M be the magnitude of a body, S its specific gravity, and W its weight, $W = MS$.

Prop. 8. When a body of uniform density floats on a fluid, the part immersed : the whole body :: specific gravity of the body : the specific gravity of the fluid.

Prop. 9. When a body is immersed in a fluid, the weight lost : whole weight of the body :: the specific gravity of the fluid : the specific gravity of the body.

Prop. 10. To describe the *hydrostatic balance*, and to shew how to find the specific gravity of a body by means of it, 1st, when its specific gravity is greater than that of the fluid in which it is weighed ; 2ndly, when it is less.

Prop. 11. To describe the common *hydrometer*, and to shew how to compare the specific gravities of two fluids by means of it.

Elastic Fluids.

Prop. 12. Air has weight.

Prop. 13. The elastic force of air at a given temperature varies as to the density.

Prop. 14. The elastic force of air is increased by an increase of temperature.

Prop. 15. To describe the construction of the common *air-pump*, and its operation.

Prop. 16. To describe the construction of the *condenser*, and its operation.

Prop. 17. To explain the construction of the common *barometer*, and to shew that the mercury is sustained in it by the pressure of the air on the surface of the mercury in the basin.

Prop. 18. The pressure of the atmosphere is accurately measured by the weight of the column of mercury in the barometer.

Prop. 19. To describe the construction of the common *pump*, and its operation.

Prop. 20. To describe the construction of the *forcing-pump*, and its operation.

Prop. 21. To explain the action of the *siphon*.

Prop. 22. To shew how to graduate a common *thermometer*.

Prop. 23. Having given the number of degrees on *Fahrenheit's* thermometer, to find the corresponding number on the *centigrade* thermometer.

Also such Questions and applications as arise directly out of the aforementioned Propositions of Mechanics and Hydrostatics *.

* Added by Grace, March 20, 1846.

EXAMINATION PAPERS.

MECHANICS AND HYDROSTATICS.

SENATE-HOUSE. FRIDAY, *Jan.* 12, 1849. 9...12.

FIRST DIVISION.

(A)

1. DEFINE force, shew how density is measured.

2. If two forces acting perpendicularly on a straight lever in opposite directions and on the same side of the fulcrum balance each other, they are inversely as their distances from the fulcrum; and the pressure on the fulcrum is equal to the difference of the forces.

3. One end of a given straight lever rests upon a fulcrum, and the other end is sustained by a force of 3 lbs. acting upwards, where must a weight of 12 lbs. be placed in order that there may be equilibrium?

4. Assuming that the resultant of two forces acting on a point lies along the diagonal of the parallelogram whose sides represent the forces in magnitude and direction, shew that it is represented in magnitude by the diagonal.

Find the magnitude and direction of the resultant of two equal forces at right angles to one another.

5. In that system of pulleys in which each pulley hangs by a separate string, shew that $P : W :: W$'s velocity : P's velocity.

6. When a body is placed on a horizontal plane, it will stand or fall, according as the vertical line, drawn from its centre of gravity, falls within or without its base.

Construct a triangle upon a given horizontal base such that the vertical line through the centre of gravity shall pass through an angle at the base.

7. Fluids press equally in all directions.

8. A given cubical vessel resting on one side in a horizontal position contains a given quantity of fluid, a body is placed in it which floats, the weight of the body being given, find the pressure on the base, and the height to which the fluid rises in the vessel.

9. When a body of uniform density floats on a fluid, the part immersed : the whole body :: the specific gravity of the body : the specific gravity of the fluid.

10. If a cubic inch of iron weigh 4½ ounces, and a cubic foot of water 1000 ounces, what is the specific gravity of iron?

11. Explain the construction of the common barometer, and shew that the mercury is sustained in it by the pressure of the air on the surface of the mercury in the basin.

12. If a barometer stands at 30 inches, what is the greatest vertical length of the suction-pipe of a common pump that will pump up mercury?

13. Shew how a common thermometer is graduated.

"Temperate" is marked on Fahrenheit's thermometer at 56°, what is its height on the centigrade?

MECHANICS AND HYDROSTATICS.

SENATE-HOUSE. FRIDAY, *Jan.* 12, 1849. 9. 12.

FIRST DIVISION.

(B)

1. DEFINE weight, and shew how a statical force is measured.

2. If two weights acting perpendicularly on a straight lever on opposite sides of the fulcrum balance each other, they are inversely as their distances from the fulcrum; and the pressure on the fulcrum is equal to their sum.

3. Two forces of 3 and 5lbs respectively act upon a given straight lever, where must the fulcrum be placed for equilibrium, supposing the forces to act in opposite directions?

4. If two forces acting on a point are represented in magnitude and direction by the two sides of a parallelogram, shew that their resultant is represented in direction by the diagonal of the paralellogram.

If three forces acting on a point will keep it at rest, shew that they will also when their insensity is doubled

5 If P and W balance each other on the inclined plane, shew that $P : W ::$ W's velocity in direction of gravity : P's velocity.

6. Find the centre of gravity of two heavy points; supposing the two points rigidly connected and a fulcrum placed under the centre of gravity, what is the pressure on the fulcrum? would the bodies be in equilibrium?

7. If a vessel, the bottom of which is horizontal and the sides vertical, be filled with fluid, the pressure upon the bottom will be equal to the weight of the fluid.

8 A crooked horn is filled with fluid, and a lid being placed over the top it is then inverted and made to rest upon its top; find the amount of pressure upon the lid

9 When a body is immersed in a fluid, the weight lost : whole weight of the body :: the specific gravity of the fluid : the specific gravity of the body.

10. A piece of wood which weighs 3lbs. and whose specific gravity . that of water :: 3 : 4 floats in water, what weight placed upon it would just sink it?

11. The pressure of the atmosphere is accurately measured by the weight of the column of mercury in the barometer

12 What difference will be produced in the action of a siphon by taking it to the top of a mountain where the barometer is 26 inches, the barometer below being at 30 inches?

13. Shew how a centigrade thermometer is graduated.

A centigrade thermometer stands at 35°, what is the height of Fahrenheit's?

MECHANICS AND HYDROSTATICS.

SENATE-HOUSE. FRIDAY, *Jan.* 12, 1849. $12\frac{1}{2}\ldots3\frac{1}{2}$.

SECOND DIVISION.

(A)

1. IF two weights acting perpendicularly on a straight lever on opposite sides of the fulcrum balance each other, they are inversely as their distances from the fulcrum; and the pressure on the fulcrum is equal to their sum

The arms of a straight lever are 12 and 18 inches respectively; and a weight of 3lbs. is suspended at the extremity of the shorter arm, what is the pressure on the fulcrum?

2. If two weights balance each other on a straight lever when it is horizontal, they will balance each other in every position of the lever.

Is the converse necessarily true? Why does the common balance not rest in all positions?

3. If three forces represented in magnitude and direction by the sides of a triangle act on a point they will keep it at rest. If two forces of 5 lbs. and 12lbs. act at right angles upon a point, find the magnitude of the force which will keep the point at rest. Find also the directions in which the two given forces must be applied, in order that the point may be kept at rest by the least possible force, and find its magnitude.

4. In a system of pulleys in which each pulley hangs by a separate string, and the strings are parallel, there is an equilibrium when $P : W :: 1$: that power of 2 whose index is the number of moveable pulleys.

5. If the weight W be supported on an inclined plane by a force P acting parallel to the plane, and the whole be put in motion, shew that $P : W ::$ W's velocity in the direction of gravity : P's velocity.

6. Find the centre of gravity of any number of heavy points, and shew that the pressure on the centre of gravity is equal to the sum of the weights in all positions.

Three weights 1 lb., 2 lbs., 3 lbs. are placed in a straight line at equal distances of 12 inches, find the distance of the common centre of gravity from the middle weight.

7. When a body is suspended from a point, it will rest with its centre of gravity in the vertical line passing through the point of suspension.

8. If a body floats in a fluid it displaces as much of the fluid as is equal in weight to the weight of the body; and it presses downwards and is pressed upwards with a force equal to the weight of the fluid displaced.

A prismatic solid whose height is five inches floats at a depth of three inches in a fluid, compare the specific gravities of the solid and fluid.

9. Describe the hydrostatic balance, and shew how to find by means of it the specific gravity of a solid lighter than the fluid in which it is weighed.

10. The elastic force of air at a given temperature varies as the density.

11. Describe the construction of the condenser and its operation.

12. Explain the construction of the common barometer, and shew that the mercury is sustained in it by the pressure of the air on the surface of the mercury in the basin.

What would be the effect 1st of a hole at the bottom of the tube; 2nd at the top?

13. Two vertical tubes are connected by a horizontal tube of 2 inches; supposing 12 inches of mercury poured into one tube, and 26 inches of water into the other; find the altitudes of the water and mercury in the two branches, the specific gravity of mercury being supposed 13 times the specific gravity of water.

MECHANICS AND HYDROSTATICS.

SENATE-HOUSE. FRIDAY, *Jan.* 12, 1849. $12\frac{1}{2} \ldots 3\frac{1}{2}$.

SECOND DIVISION.

(B)

1. If two forces acting perpendicularly on a straight lever in opposite directions and on the same side of the fulcrum balance each other, they are inversely as their distances from the fulcrum; and the pressure on the fulcrum is equal to the difference of the forces.

If the arms of the lever are 12 and 18 inches respectively, and a weight of 4lbs. is suspended at the extremity of the longer arm, what is the magnitude and direction of the pressure on the fulcrum?

2. If two forces acting at any angles on the arms of any lever balance each other, they are inversely as the perpendiculars drawn from the fulcrum to the directions in which the forces act.

P and Q are two forces whose directions make equal angles with the arms of a bent lever; the lengths of the arms are 6 and 8 inches respectively; find the relation between P and Q when they balance each other.

3. If three equal forces act upon a point and keep it at rest, find the inclinations of their directions to each other. Find also the directions in which three forces represented by 3lbs., 5lbs. and 8lbs. must be applied to a point so as to keep it in equilibrium.

4. In a system of pulleys in which the same string passes round any number of pulleys, and the parts of it between the pulleys are parallel, there is an equilibrium when the power : the weight :: 1 : the number of strings at the lower block.

5. Shew that if P and W balance each other on the wheel and axle and the whole be put in motion, $P : W :: W$'s velocity : P's velocity.

6. If a body balance itself on a line in all positions, the centre of gravity is in that line. If a body balance itself on a line in a certain position, what will be the position of the centre of gravity?

7. Find the centre of gravity of a triangle, and also of three equal weights placed at three angular points and shew that they coincide.

8. The surface of every fluid at rest is horizontal.

9. Define a fluid, and prove that the pressure upon any particle of fluid of uniform density is proportional to its depth below the surface of the fluid. Two vessels are filled with fluid and placed upon a horizontal plane. The bases are 1 square foot and 2 square feet respectively, and altitudes 9 and 6 inches, compare the pressures upon the bases of the vessels.

10. When a body is immersed in a fluid, the weight lost : whole weight :: the specific gravity of the body : the specific gravity of the fluid.

If the specific gravities of iron and gold be 8 and 19 times the specific gravity of water respectively; find the weight in water of a substance combined of 1 lb. of iron and 1 lb. of gold.

11. A weight of 4lbs. when placed upon a piece of wood whose specific gravity : that of water :: 3 : 5 just causes it to sink; find the weight of the wood.

12. Describe the construction of the common air-pump, and its operation.

13. The pressure of the atmosphere is accurately measured by the weight of the column of mercury in the barometer.

If 13 inches of water be inserted in the tube upon the mercury, what will be the altitude of the upper surface of the water when the common barometer stands at 30 inches, the specific gravity of mercury being supposed 13 times that of water? How much will the top of the water fall, when the mercurial barometer sinks an inch?

APPENDIX.

BOOK III.

THE LAWS OF MOTION.

REMARKS

ON

MATHEMATICAL REASONING

AND THE

LOGIC OF INDUCTION.

BOOK III.

THE LAWS OF MOTION.

DEFINITIONS AND FUNDAMENTAL PRINCIPLES.

1. The science which treats of Force producing Motion, and of the Laws of the Motion produced, is Dynamics.

2. In Dynamics, we adopt the Ideas, Definitions, Axioms, and Propositions of Statics.

3. We require also several new Ideas, Definitions, and Principles, which are obtained by Induction, and will be stated in the succeeding Propositions.

4. Velocity is the degree in which a body moves quickly or slowly: thus, if a body describes a greater space than another in the same time, it has a greater velocity.

5. The velocity of a body is *uniform* when it describes equal spaces in *all* equal times.

6. The velocities of bodies, when uniform, are *as* the spaces which they describe in equal times.

Def. 1. The velocity of a body moving uniformly is *measured* by the space described in a unit of time.

When the velocities of bodies are not uniform, they are increasing or decreasing.

Axiom 1. If a body move with an *increasing* velocity, the space described in any time is *greater* than

the space which would have been described in the same time, if the velocity had continued uniform for the same time and the same as it was at the *beginning* of that time.

And the space described in any time is *less* than the space which would have been described in the same time, if the velocity had been uniform for the same time and the same as it is at the *end* of that time.

Axiom 2. If a body move with a *decreasing* velocity the above Axiom is true, putting "less" for "greater," and "greater" for "less."

Axiom 3. If two bodies move, having their velocities at every instant in a constant ratio, the space described in any time by one body and by the other will be in the same ratio.

Axiom 4. If several detached material points, acted upon by any forces, move in parallel lines, parallel to the forces, in such a manner as to retain always the same distances from each other, and the same relative positions; they may be supposed to be rigidly connected, and acted upon by the same forces, and their motions will not be altered on this supposition.

Axiom 5. On the same supposition, the parallel forces may be supposed to be added together so as to become one force, and the motions will not be altered.

Axiom 6. When bodies in motion exert pressure upon each other, by means of strings, rods, or in any other way, the reaction is equal and opposite to the action at each point.

Definition 2 (of Force), Def. 3 (of the Direction of Force), stand after Prop. 3; Def. 4 (of Uniform Force), stands after Prop. 3; Def. 5 (of Composition

of Motions), after Prop. 8; Def. 6 (of Accelerating Force), after Prop. 13; Def. 7 (of Momentum), Def. 8 (of Elastic and Inelastic Bodies), Def. 9 (of Direct Impact) after Prop. 17 of this Book.

Axiom 7 stands after Prop. 2; Axiom 8 and 9 after Prop. 17 of this Book.

Prop. I. In uniform motion, the space described with a velocity v in a time t is tv.

For (Def. 1.) v is the space described in each unit of time, and t the number of units; therefore the whole space described is tv.

Prop. II. *Inductive Principle I. First Law of Motion.*

A body in motion, not acted upon by any force, will go on for ever with a uniform velocity.

The facts which are included in this induction are such as the following:—

(1) All motions which we produce, as the motions of a body thrown along the ground, of a wheel revolving freely, go on for a certain time and then stop.

(2) Bodies falling downwards go on moving quicker and quicker as they fall farther.

It was attempted to explain these facts, by saying that motions such as (1) are *forced* motions, and motions such as (2) are *natural* motions; and that forced motions decay and cease by their nature, while natural motions, by their nature, increase and become stronger.

But this explanation was found to be untenable; for it was seen—(3) that forced motions decayed less and less by diminishing the obvious obstacles. Thus a

body thrown along the ground goes farther as we diminish the roughness of the surface; it goes farther and farther as the ground is smoother, and farther still on a sheet of ice. The wheel revolves longer as we diminish the roughness of the axis; and longer still, if we diminish the resistance of the air by putting the wheel in an exhausted receiver.

Thus a decay of the motion in the cases (1) is constantly produced by the obstacles. Also an increase of the motion in the cases (2) is constantly produced by the weight of the body.

Therefore there is in these facts nothing to show that any motion decays or increases by its nature, independent of the action of external causes.

(3) By more exact experiments, and by further diminishing the obstacles, the decay of motion was found to be less and less; and there was in no case any remaining decay of motion which was not capable of being ascribed to the remaining obstacles.

Hence the facts are explained by introducing the *Idea of force*, as that which causes change in the motion of a body; and the *Principle*, that when a body is not acted upon by any force, it will move with a uniform velocity.

Cor. 1. When a body moves freely (not being retained by any axis or any other restraint), and is not acted upon by any force, it will move in a straight line.

For since it is not acted on by any force, there is nothing to cause it to deviate from the straight line on any one side.

Def. 2. *Force* is that which causes change in the state of rest or motion of a body.

Def. 3. When a Force acts upon a body, and puts it in motion, the line of direction of the motion is the *direction of the force.*

Axiom 7. When a Force acts upon a body in motion, so that the direction of the force is the direction of the motion, the force will not alter the direction of the motion.

Prop. III. *Inductive Principle VI. Gravity is a uniform force.*

The facts which are included in this induction are such as the following:—

(1). Bodies falling directly downwards fall quicker and quicker as they descend.

It was inferred, as we have seen in the last Proposition, that the additions of velocity in the falling bodies are caused by gravity.

An attempt was made to assign the law of the increase of velocity conjecturally, by introducing the Definition, that a uniform force is a force which, acting in the direction of a body's motion, adds equal velocities in equal *spaces*, and the Proposition that gravity is a uniform force.

The Definition is self-contradictory. But if it had not been so, the Proposition could only have been confirmed by experiment.

(2). It appeared by experiment that when bodies fall (down inclined planes) the spaces described are as the squares of the times from the beginning of the motion.

This was distinctly explained and rigorously deduced by introducing the *Definition* of *uniform force;* that it is a force which, acting in the direction of the body's motion, adds equal velocities in equal *times;*

And the *Principle* that gravity (on inclined planes) is a uniform force.

For it may be proved deductively, as we shall see, that this definition being taken, the spaces described in consequence of the action of a uniform force are as the squares of the times from the beginning of the motion. And if the force be other than uniform, the spaces will not follow this law. Therefore the Proposition, that gravity on inclined planes is a uniform force, is the only one which will account for the results of experiment.

Also if the force of gravity on inclined planes be a uniform force, the force of gravity when bodies fall freely is uniform, for when the inclined plane becomes vertical, the law must remain the same.

(3). The Proposition is further confirmed by shewing that its results, obtained deductively, agree with experiments made upon two bodies which draw each other over a fixed pulley (Atwood's Machine); and—(4) by the times of oscillation of pendulums.

Also it appears that when gravity acts in a direction opposite to that of a body's motion, it subtracts equal velocities in equal times.

Hence we introduce the following Definition.

Def. 4. A *uniform force* is that which, acting in the direction of the body's motion, adds (and in the opposite direction, subtracts,) equal velocities in equal times.

Prop. IV. If a uniform force act upon a body, moving it from rest, and if a be the velocity at the end of a unit of time, v, the velocity at the end of t units of time, is ta.

For the body will move in the direction of the force (Def. 3), and therefore the force is in the direction of the motion; and therefore by Axiom 7, the direction of the motion is not altered by the action of the force. Hence by Def. 4, the velocity

added to the velocity in each second is a, and in t seconds from the beginning of the motion it is ta.

Cor. 1. At the end of $\frac{1}{n}$ of a unit of time the velocity is $\frac{a}{n}$.

Cor. 2. At the end of $\frac{m}{n}$ units of time, the velocity is $\frac{ma}{n}$.

Cor. 3. If v be the velocity at the end of the time t, the velocity at the end of the time $\frac{m}{n}t$ will be $\frac{m}{n}v$.

Prop. V. If a uniform force act upon a body moving it from rest, and if a be the velocity at the end of a unit of time, s, the space described at the end of t units of time, is $\frac{1}{2}at^2$.

Let each unit of time be divided into n equal portions; each of these will be $\frac{1}{n}$; and the whole number will be tn; and the velocity at the beginning of the first, second, third, fourth, &c. of these portions will be, by Prop. 1, Cor. 2,

$$0, \frac{a}{n}, \quad 2a \quad 3a \quad , \text{\&c. } (tn \text{ terms}).$$

Suppose spaces to be described in these portions of time with the velocity at the beginning of each portion continued uniform during that portion; these spaces are by Prop. 1,

$$0 \times \frac{1}{n}, \ \frac{a}{n} \times \frac{1}{n}, \ \frac{2a}{n} \times \frac{1}{n}, \ \frac{a}{n} \times \frac{1}{n} \ (tn \text{ terms})$$

which form an arithmetical series. And the last term of this series is

$$\frac{(tn - 1)\,a}{n} \times \frac{1}{n}.$$

And the sum of it is (Introd. Art. 60)

$$\frac{(tn - 1)\,a}{n} \times \frac{1}{n} \times \frac{tn}{2};$$

$$\text{or } \frac{(tn - 1)\,at}{2n} \text{ or } \frac{at^2}{2} - \frac{at}{2n}.$$

In the same manner the velocity at the *end* of the first, second, third, &c. of these portions is

$$\frac{a}{n},\ \frac{2a}{n},\ \frac{3a}{n},\ \&\text{c. } (tn \text{ terms}).$$

Suppose spaces to be described in these portions of time with the velocity at the end of each portion continued uniform during the time. These are as before

$$\frac{a}{n} \times \frac{1}{n},\ \frac{2a}{n} \times \frac{1}{n},\ \frac{3a}{n} \times \frac{1}{n},\ \frac{4a}{n} \times \frac{1}{n}\ (tn \text{ terms});$$

an arithmetical progression, of which the last term is $\frac{tna}{n} \times \frac{1}{n}$, and the sum is $\left(\frac{tna}{n} \times \frac{1}{n} + \frac{a}{n} \times \frac{1}{n}\right)\frac{tn}{2}$, or $\frac{(tn + 1)\,at}{2n}$ or $\frac{at^2}{2} + \frac{at}{2n}$.

But in this case the body moves with a constantly increasing velocity. Therefore by Axiom 1, the space described in each of the times $\frac{1}{n}$ is greater than the space described on the former of the above suppositions; and less than the space described on the latter of the above suppositions. Hence the whole space s is greater than $\frac{at^2}{2} - \frac{at}{2n}$, and less than $\frac{at^2}{2} + \frac{at}{2n}$.

Therefore it is equal to $\frac{at^2}{2}$; for if not, let it be equal to a greater quantity, as $\frac{at^2}{2} + b$, and let $n = \frac{at}{2b}$: then $\frac{at}{2n} = b$; and therefore the space described is equal to $\frac{at^2}{2} + \frac{at}{2n}$. but it is less than this; which is impossible. Therefore the space is not equal to a greater quantity than $\frac{at^2}{2}$; and in like manner it may be shewn that it is not equal to a less. Therefore the space is equal to $\frac{at^2}{2}$. Q.E.D.

COR. Hence if t, T, be any two times from the beginning of the motion and s, S the spaces through which a body falls in those times, $s : S :: t^2 : T^2$.

PROP. VI. **The space described in any time from rest by the action of a uniform force is equal to half the space described by the last acquired velocity continued uniform for the time.**

As in last Proposition, let t be the whole time, and a the velocity acquired in one unit of time. Then at is the velocity acquired in the whole time t. And a body moving with this velocity for the time t would describe the space at^2 by Prop. 1. But a body moving from rest by a uniform force describes the space $\frac{1}{2}at^2$ by Prop. 5. Therefore the latter space is half the former. Q.E.D.

COR. 1. A body falling from rest by the uniform force of gravity, describes 16 feet in one second.

Therefore with the velocity acquired it would describe 32 feet in one second. Therefore gravity generates a velocity of 32 feet in one second of time.

Cor. 2. If g represent 32 feet, the space through which a body falls in t seconds by the action of gravity is $\frac{1}{2}gt^2$.

Prop. VII. When a body is projected in a direction opposite to the direction of a uniform force, with a velocity v, the whole time (t) of its motion till its velocity is destroyed, and the space (s) described in that time, are known by the equations $v = at$, $s = \frac{1}{2}at^2$.

For by the Definition of uniform force, the force, acting in a direction opposite to the motion, subtracts in equal times the same velocities which the same force adds when it acts in the direction of the motion. Therefore at a series of units of time the velocities will be v, $v - a$, $v - 2a$, $v - ta$, till $v - ta$ becomes 0, when all the velocity is destroyed; and when this occurs, $v - ta = 0$, or $v = ta$.

Also by Ax. 2, the same reasoning would hold as in Prop. 5, putting less for greater and greater for less; and therefore the same conclusion is true, namely, $s = \frac{1}{2}at^2$.

Prop. VIII. *Inductive Principle III. Second Law of Motion.* When any force acts upon a body in motion, the motion which the force would produce in the body at rest is compounded with the previous motion of the body.

The facts which the Induction includes are, in the first place, such as the following:—

(1). A stone dropped by a person in motion, is soon left behind him.

From (1) it was inferred that if the earth were in motion, bodies dropt or thrown would be left behind.

But it appeared that the stone was not left behind while it was moving in free space, and was only stopt when it came to the ground. Again it was found by experiment,

(2). That a stone dropt by a person in motion describes such a path that, relatively to him, it falls vertically.

(3). A man throwing objects and catching them again uses the same effort whether he be at rest or in motion.

Again, such facts as the following were considered:

(4). A stone thrown horizontally or obliquely describes a bent path and comes to the ground.

It was at first supposed that the stone does not fall to the ground till the original velocity is expended. But when the First Law of Motion was understood, it was seen that the gravity of the stone must, from the first, produce a change in the motion, and deflect the stone from the line in which it was thrown. And by more exact examination it appeared that (making allowance for the resistance of the air),—(5) the stone falls below the line of projection by exactly the space through which gravity in the same time would draw it from rest.

These facts were distinctly explained and rigorously deduced by introducing the *Definition* of *Composition of Motions*;—that two motions are compounded

when each produces its separate effect in a direction parallel to itself;

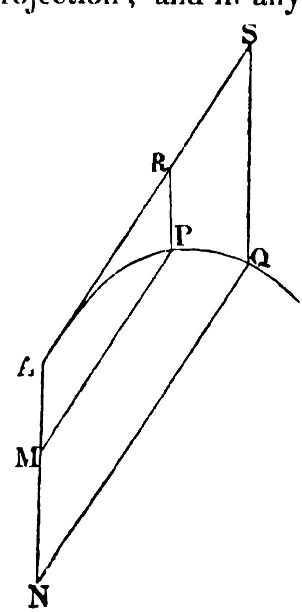
And the *Principle*, that when a force acts upon a body in motion, the motion which the force would produce in the body at rest is compounded with the previous motion of the body.

The Proposition is confirmed by shewing that its results, deduced by demonstration, agree with the facts.

DEF. 5. Two *motions* are *compounded* when each produces its separate effect in a direction parallel to itself.

PROP. IX. If a body be projected in any direction and acted upon by gravity, in any time it will describe a curve line of which, the tangent intercepted by the vertical line, and the vertical distance from the tangent, are respectively the spaces due to the original velocity and to the action of gravity in that time.

Let *AR* be the direction of projection; and in any time, let *AR* be the space which the body would have described with the velocity of projection in that time, and *AM* the space through which the body would have fallen in the same time. Then, completing the parallelogram *AMPR*, the body will, by the Second Law of Motion (Prop. 8) be found at *P*, and *RP* is equal to *AM*. Also *AR* is a tangent to the curve at *A*, because at *A* the body is moving in the direction *AR*. Therefore, &c. Q.E.D.



COR. If P, Q be the points at which the projectile is found, at any two times t, T from its being at A, and if PR, QS be vertical lines, meeting the tangent at A in R, S, then

$$PR : QS :: AR^2 : AS^2.$$

For $PR : QS :: t^2 : T^2$ by Cor. to Prop. 5.

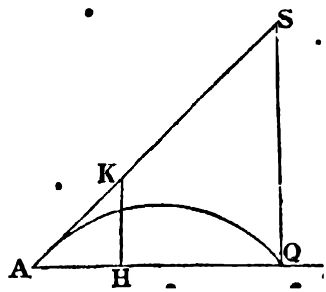
But $t : T :: AR : AS$; whence

$$t^2 : T^2 :: AR^2 : AS^2.$$

Therefore $PR : QS :: AR^2 : AS^2$.

PROP. X. A body is projected from a given point in a given direction; to find the range upon a horizontal plane, and the time of flight.

The *range* is the distance from the point of projection to the point where the *projectile* (or body projected) again strikes a plane passing through the point of projection.



Let a body be projected in a direction AK, such that, AH, HK being horizontal and vertical, $AH : HK :: m : n$. Hence

$$\frac{AH}{HK} = \frac{m}{n}, \quad \frac{AK^2}{HK^2} = 1 + \frac{AH^2}{HK^2} = 1 + \frac{m^2}{n^2} = \frac{n^2 + m^2}{n^2}$$

$$\frac{HK}{AK} = \frac{n}{\sqrt{n^2 + m^2}} \qquad \frac{AH}{AK} = \frac{m}{n} \cdot \frac{HK}{AK} = \frac{m}{\sqrt{n^2 + m^2}}.$$

Let v be the velocity of projection, AQ the path described, QS vertical; and let the time of describing AQ be t. Therefore, by the last Proposition, AS, the space described with velocity v in the time t, will be vt. Also SQ, the space fallen by gravity in the time t, will be $\frac{1}{2}gt^2$, by Prop. 6, Cor. 2.

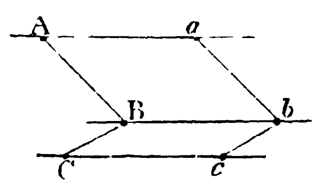
And $\dfrac{SQ}{AS} = \dfrac{HK}{AK}$; that is, $\dfrac{\frac{1}{2}gt^2}{vt} = \dfrac{n}{\sqrt{n^2 + m^2}}$,

$\dfrac{gt}{2v} = \dfrac{n}{\sqrt{n^2 + m^2}}$, $t = \dfrac{2v}{g} \cdot \dfrac{n}{\sqrt{m^2 + n^2}}$; which is the time of flight.

Also $\dfrac{AQ}{AS} = \dfrac{AH}{AK}$, or $\dfrac{AQ}{vt} = \dfrac{m}{\sqrt{m^2 + n^2}}$, $AQ =$ $vt\dfrac{m}{\sqrt{m^2 + n^2}}$; $AQ = \dfrac{2v^2}{g} \dfrac{mn}{m^2 + n^2}$, which is the range.

PROP. XI. If any particles, moving in parallel directions, and acted upon each by a certain force in the direction of its motion, move with velocities which are equal for all the particles at every instant, the motions of the particles will be the same if we suppose them to be connected so as to form a single rigid body, and the forces to be added together so as to form a single force.

Let A, B, C, be any particles acted upon by any forces, and moving in parallel directions with velocities which are equal at every instant. Since the velocities at every instant are equal, the spaces described in the same time are equal for all the particles, by Axiom 3.



Let Aa, Bb, Cc be the spaces described in any time, which are therefore equal and parallel. Therefore ab is equal and parallel to AB, and bc to BC, and so on. Therefore the relative positions and dis-

tances of the particles A, B, C are not altered by their motion into the places a, b, c.

Therefore, by Axiom 4, if we suppose the particles A, B, C to be rigidly connected, their motions will not be altered; that is, the motions will not be altered if A, B, C are supposed to be particles of a single rigid body.

Also, by Axiom 5, if we suppose the forces which act upon the particles, A, B, C, to be added together so as to form a single force, the motion will not be altered.

Therefore, &c. Q.E.D.

PROP. XII. If, on two bodies, two pressures act, which are proportional to the quantities of matter in the two bodies, the velocities produced in equal times in the two bodies are equal.

Let P, Q, be two pressures, and m, n two bodies, measured by the number of units of quantity of matter which each contains; and let $P : Q :: m : n$.

Let the pressure P be divided into m parts, each of which will be $\frac{P}{m}$, and let each of these parts of the force act upon a separate one of the m units into which the body m can be divided, and let it produce in a time t a velocity v. Each of the pressures $\frac{P}{m}$ will produce in the unit upon which it acts for the time t, an equal velocity v, in a direction parallel to P. Therefore, if all the m pressures act for the same time t upon the m units of the body respectively, all the units will move with velocities which are equal at every instant. Therefore, by Prop. 11, if we suppose the m units to be connected so as to form one rigid

body m, and the forces to be added so as to form a single force P, the motion will still be the same. That is, the pressure P acting upon the body m, will produce the velocity v in the time t.

In the same manner it may be shewn that the pressure Q acting upon the body n will produce the same velocity which a pressure $\frac{Q}{n}$ produces in a body 1.

But since $P : Q :: m : n$, $\frac{Q}{n} = \frac{P}{m}$; therefore $\frac{Q}{n}$ acting upon a body 1 will produce a velocity v in a time t. Therefore Q acting on n will produce a velocity v in a time t; the same which P produces in m. Q.E.D.

PROP. XIII. *INDUCTIVE PRINCIPLE IV. The Third Law of Motion.* When pressure generates (or destroys) motion in a given body, the accelerating force is as the pressure.

The facts included in this Induction are such as the following:—

(1). When pressure produces motion, the velocity produced is greater when the pressure is greater.

In order to determine in what proportion the velocity increases with the pressure, further consideration and inquiry are necessary.

It appeared that,

(2). On an inclined plane the velocity acquired by falling down the plane is the same as that acquired by falling freely down the vertical height of the plane (Galileo's experiment).

(3). When two bodies P, Q* hang over a fixed pulley, the heavier P descends, and the velocity gene-

* See figure to Prop. 17.

rated in a given time is as $P - Q$ directly, and as $P+Q$ inversely (Atwood's Machine).

(4). The small oscillations of pendulums are performed in times which are as the square roots of the lengths of the pendulums.

(5). In the impacts of bodies the momentum gained by the one body is equal to the momentum lost by the other (Newton's Experiments).

(6). In the mutual attractions of bodies the center of gravity remains at rest.

These results are distinctly explained and rigorously deduced by introducing the *Definition* of uniform Accelerating Force;—that it is as the velocity generated (or destroyed) in a given time;

And the *Principle* that the Accelerating Force for a given body is as the pressure.

Most of these consequences will be proved in the succeeding Propositions, (14, 15, 16, 17, 18), and thus this Inductive Proposition is confirmed.

DEF. 6. Uniform Accelerating Force is *measured* by the velocity generated in a unit of matter in a unit of time.

Hence in the formula in Prop. 4 and 5, a represents the Accelerating Force.

AXIOM 8. If two bodies move so that their velocities at every instant are equal, the Accelerating Forces of the two bodies at every instant are equal; and conversely.

AXIOM 9. If two bodies move so that their Accelerating Forces at every instant are in a constant ratio, and are in the direction of the motion, the velocities added or subtracted in any time are in the ratio of the Accelerating Forces.

Prop. XIV. In different bodies, the Accelerating Force is as the pressure which produces motion directly, and as the quantity of matter moved inversely.

Let two pressures P, Q, produce motion in two bodies of which the quantities of matter are M, N. Let $M : N :: P : X$; therefore, by Prop. 12, the force X would, in a given time, produce in N the same velocity which P would produce in M; that is, the Accelerating Force on M arising from the pressure P, is equal to the Accelerating Force on N arising from the pressure X.

But by the Third Law of Motion (Prop. 13) the Accelerating Force on N arising from the pressure X is to the Accelerating Force on the same body N arising from the pressure Q as X is to Q.

Therefore, the Accelerating Force on M arising from P is to the Accelerating Force on N arising from Q as X is to Q.

But $M : N :: P : X$; therefore $X = \frac{PN}{M}$, and therefore X is to Q, as $\frac{PN}{M}$ is to Q, or as $\frac{P}{M}$ to $\frac{Q}{N}$.

Therefore the Accelerating Forces of P on M and of Q on N are as $\frac{P}{M}$ and $\frac{Q}{N}$. Q.E.D.

Prop. XV. On the inclined plane, the time of falling down the plane is to the time of falling freely down the vertical height of the plane as the length of the plane to its height.

Let L be the length of the plane, H its height. The pressure which urges a body down an inclined

plane is equal to the pressure which would support it acting in the opposite direction; but this pressure : W, the weight of the body :: $H : L$ (B. I. Prop. 20.) Therefore the pressure which produces motion on the plane is $\frac{WH}{L}$.

The quantity of matter of the body is as W.

Hence, since by the last Proposition the Accelerating Force on the inclined plane is as the pressure directly and the quantity of matter inversely; therefore Accelerating Force on Inclined Plane : Accelerating Force of body falling freely :: $\frac{WH}{WL} : \frac{W}{W}$ :: $H : L$.

Now the force on the inclined plane is a uniform accelerating force; and therefore the velocity acquired in a unit of time measures it, by Def. 6. Therefore, if La be the velocity acquired in a unit of time by a body falling freely, Ha will be the velocity acquired in a unit of time down the inclined plane. And the rule of Prop. 5 is applicable. Therefore, if t and t' be the times of falling down L, and of falling vertically down H,

$$L : H :: \tfrac{1}{2}Hat^2 : \tfrac{1}{2}Lat'^2;$$
$$\text{or } L^2 : H^2 :: t^2 : t'^2;$$
$$\text{or } L : H :: t : t'.$$

Prop. XVI. On the inclined plane, the velocity acquired by falling down the inclined plane is equal to the velocity acquired by falling freely down the vertical height of the plane.

As before, the accelerating force on the plane is to the accelerating force of a body falling freely

$$:: H : L;$$

Also $s = \frac{1}{2} vt$, by Prop. 6; whence as before, v' being the velocity acquired by falling freely down H,

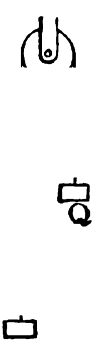
$$L : H :: \tfrac{1}{2}vt : \tfrac{1}{2}v't' :: vt : v't';$$

$$\text{But, } H : L \qquad :: t' : t \text{ by Prop. 15.}$$

$$\text{Therefore } 1 : 1 \qquad :: v : v';$$

Whence $v = v'$; the velocity acquired down the plane is equal to the velocity acquired down the vertical height. Q.E.D.

PROP. XVII. When two bodies P, Q hang over a fixed pulley, and move by their own weight merely*, the heavier P descends, and the lighter Q ascends, by the action of an accelerating force which is as $\dfrac{P-Q}{P+Q}$.



The string which connects P and Q exerts an equal pressure in opposite directions upon P and upon Q, (Axiom 6). Let this pressure be X. Then since P is urged downwards by a force P and upwards by a force X, it is on the whole urged downwards by a force $P - X$. And the quantity of matter is P. Therefore, by Prop. 14, the Accelerating Force upon P downwards is as $\dfrac{P-X}{P}$. In the same manner, since X acts upwards upon Q and the weight of Q acts downwards, the accelerating force upon Q upwards is as $\dfrac{X-Q}{Q}$. But the accelerating force upon Q upwards and upon P downwards must be equal, because they move at every point with equal velocities, by Axiom 8.

* That is, neglecting the effect of the matter in the pulley and the string.

Therefore $\frac{X-Q}{Q}$ is equal to $\frac{P-X}{P}$.

that is, $\frac{X}{Q} - 1$ is equal to $1 - \frac{X}{P}$;

or $\frac{X}{Q} + \frac{X}{P}$ is equal to 2.

Therefore $\frac{X(P+Q)}{PQ}$ is equal to 2;

and X is equal to $\frac{2PQ}{P+Q}$.

Hence $P - X$ is $P - \frac{2PQ}{P+Q}$, or $\frac{P^2 - PQ}{P+Q}$; and the accelerating force upon P, which is as $\frac{P-X}{P}$, is as $\frac{P-Q}{P+Q}$. And, in like manner, the accelerating force upon Q is as $\frac{P-Q}{P+Q}$.

DEF. 7. The *momentum* of a body is the product of the numbers which express its velocity and its quantity of matter.

DEF. 8. *Elastic* bodies are those which separate when one impinges upon another; *inelastic* bodies are those which do not separate after impact.

DEF. 9. The impact of two bodies is *direct*, when the bodies, before impact, either moving in the same direction or one of them being at rest, the pressure which each exerts upon the other is in the direction of the motion.

PROP. XVIII. **In the direct impact of two bodies the momenta gained and lost are equal.**

Let P impinge upon Q directly, and let X be the pressure which each exerts upon the other at any instant. Therefore the accelerating forces which act upon P and Q in opposite directions are as $\frac{X}{P}$ and $\frac{X}{Q}$; and are therefore at every instant in the constant ratio of $\frac{1}{P}$ to $\frac{1}{Q}$, or of Q to P. Therefore, by Ax. 9, the velocities generated in Q and destroyed in P, in any time, are in the same constant ratio of Q to P. And the quantities of matter are as P and Q. Therefore, by Def. 7, the momentum generated in Q and the momentum destroyed in P, in any time, are as PQ to PQ; that is, they are equal. Q.E.D.

COR. 1. If P and Q are elastic, they will separate after the impact; and the momenta generated and destroyed in Q and P by the elasticity will still be equal, for the same reasons as before.

COR. 2. The velocity destroyed in P, according to Cor. 1, may be greater than its whole velocity. In this case, P will, after the impact, move in the opposite direction with a velocity which is the excess of the velocity lost over the original velocity.

COR. 3. Before the impact, Q may move in a direction opposite to P. In this case the velocity gained by Q is to be understood as including the velocity in the opposite direction, which is destroyed.

COR. 4. If two bodies P and Q, move in opposite directions with velocities which are in the ratio of Q to P, they will be at rest after impact if they are inelastic. For since they are inelastic, they will not

separate after impact: therefore they will either be at rest or move on together. But if they move in the direction of P's motion, P has lost less than its whole velocity, and Q has gained more than its own velocity. But this is impossible, for the velocities lost and gained are in the ratio of Q to P; that is, in the ratio of P's velocity to Q's velocity. Therefore the bodies do not move in the direction of P's motion. And, in like manner, it may be shown that they do not move in the direction of Q's motion. Therefore they remain at rest.

PROP. XIX. The mutual pressure, attraction, or repulsion, or direct impact of two bodies, cannot put in motion their center of gravity.

Let two bodies P, Q, act upon each other by pressure, attraction, or repulsion, the force which each exerts upon the other (Axiom 6) being X. Therefore (Prop. 14) the accelerating forces which act on P and Q are as $\frac{X}{P}$ and $\frac{X}{Q}$ respectively, or in the constant ratio of Q to P. Therefore the velocities acquired by P and Q in any equal times are in this ratio by Axiom 9, and therefore the spaces are in the same ratio by Axiom 3.

Let P, Q, be any two bodies of which the center of gravity is C, which is at first at rest. Therefore by B. I, Prop. 24, $Q : P :: CP : CQ$, and $CQ = \frac{P}{Q} CP$. And if Pp, Qq be any spaces described in equal times, by the mutual pressure, attraction, or repulsion of the bodies, it has been proved that $Q : P$

$:: Pp : Qq$; and therefore $Qq = \frac{P}{Q} Pp$. Hence, subtracting, it follows that $Cq = \frac{P}{Q} Cp$, or $Q : P :: Cp : Cq$. And therefore C is still the center of gravity of the bodies P, Q, when they are come into the positions p, q; that is, the center of gravity has not been put in motion.

Also if the two bodies P, Q, not attracting or repelling each other, move towards each other with uniform velocities which are in the ratio of Q to P, and impinge; the spaces described in any time (as Pp, Qq) will be in the same ratio of Q to P, and, as above, the center of gravity will be at rest. And when the bodies impinge on each other, the velocities of each will either be destroyed, or destroyed and generated in an opposite direction; and in either case, since the mutual pressure is equal on both, the accelerating forces which destroy and generate velocity, will be in the ratio of Q to P, as in Prop. 17. Therefore the velocities destroyed and generated are in the same ratio as the original velocities. Therefore if the whole velocity of one body is destroyed, the whole velocity of the other body also is destroyed, and the bodies are both at rest, and their center of gravity is still at rest after impact.

But if the velocities be destroyed, and velocities generated in an opposite direction, these new velocities will also be in the ratio of the original velocities, because the accelerating forces at every instant are so, (Ax. 9); and therefore the spaces described in any time by the new velocities will be in the same ratio; and therefore, as before, it may be shown that C is still the center of gravity of P, Q.

Therefore, under all the circumstances stated, the center of gravity remains at rest. Q.E.D.

Examples to Propositions 4, 5, 6, 7; 10, 17, 18.

By means of these Propositions, we can solve such Examples as the following:—

When a body falls freely by the action of gravity, the quantity a in Prop. 4 is 32 feet, the unit of time being one second, and $v = gt$. Also (Prop. 6, Cor. 2) $v = \frac{1}{2}gt^2$.

Ex. 1. To find the velocity acquired by a body which falls by gravity for 30 seconds.

$$v = gt = 32 \times 30 = 960 \text{ feet per second.}$$

2. To find the space fallen through in the same time, $s = \frac{1}{2}gt^2 = 16 \times 30^2 = 14400$ feet.

3. To find in what time a body falls through 1024 feet.

$$1024 = 16 \times t^2,\ t^2 = 64,\ t = 8 \text{ seconds.}$$

4. To find the velocity acquired in the same space, $v = gt = 32 \times 8 = 256$ feet per second.

5. A body is projected directly upwards, with a velocity of 1000 feet a second; how high will it go?

By Prop. 7, the height will be that through which a body must fall to acquire the same velocity.

Now since

$$v = gt,\ 1000 = 32t,\ t = \frac{1000}{32} = \frac{125}{4} = 31\tfrac{1}{4}''.$$

$$s = \tfrac{1}{2}gt^2 = 16\frac{(125)^2}{4^2} = (125)^2 = 15625 \text{ feet.}$$

6. A body is projected with a velocity of 32 feet a second in a direction which makes with the horizon half a right angle: to find the flight and the range.

In this case $m = n$; therefore, by Prop. 10,

$$\frac{n}{\sqrt{n^2+m^2}} = \frac{1}{\sqrt{2}}; \; t = \frac{2v}{g} \cdot \frac{1}{\sqrt{2}} = \frac{2 \times 32}{32 \times \sqrt{2}}$$

$$= \sqrt{2} = 1.4 \text{ seconds};$$

$$\text{the range} = \frac{2v^2}{g} \frac{mn}{m^2+n^2} = \frac{2 \times (32)^2}{32} \frac{1}{2} = 32 \text{ feet}.$$

7. A cannon ball is projected with a velocity of 1600 feet a second, in a direction which rises 3 feet in 4 feet horizontal: find the time of flight and the range.

$$\frac{n}{\sqrt{n^2+m^2}} = \frac{3}{\sqrt{9+16}} = \frac{3}{5}; \; t = \frac{2 \times 1600}{32} \times \frac{3}{5}$$

$$= 75 \text{ seconds}$$

$$\text{the range} = \frac{2v^2}{g} \frac{nm}{m^2+n^2} = \frac{2(1600)^2}{32} \times \frac{12}{25}$$

$$= 76800 \text{ feet}.$$

8. An inelastic body A impinges directly on another inelastic body B at rest, with a velocity of 10 feet a second; A being 3 and B 2 ounces, find the velocity after impact.

If x be the velocity of both after impact, the velocity lost by A is $10 - x$, and the velocity gained by B is x. Hence the momentum lost by A is $3 \times (10 - x)$; and that gained by B is $2 \times x$; and these are equal by Prop. 18; therefore

$$3(10 - x) = 2x, \; 30 = 3x + 2x, \; x = 6.$$

9. The bodies being perfectly elastic, find the motions after impact.

In perfectly elastic bodies, the velocity lost by A and the velocity gained by B in the restitution of the figure are equal to the velocity lost by A and gained by B in the compression.

Now the velocity lost by A in the compression is $10 - 6$ or 4; therefore the whole velocity lost by A is 8, and its remaining velocity 2.

And the velocity gained by B in the compression is 6, and therefore the whole velocity gained by B is 12, which is B's velocity after impact.

10. A body A (3 ounces) draws B (2 ounces) over a fixed pulley: find the space described in one second from rest.

By Prop. 17, the accelerating force is as $\frac{3-2}{3+2}$; that is, it is $\frac{1}{5}$ of gravity; and the space in a second is as the force: therefore the space described in one second is $\frac{16}{5}$, or $3\frac{1}{5}$ feet.

REMARKS

ON

MATHEMATICAL REASONING,

AND ON

THE LOGIC OF INDUCTION.

SECT. I. *On the Grounds of Mathematical Reasoning.*

1. THE study of a science, treated according to a rigorous system of mathematical reasoning, is useful, not only on account of the positive knowledge which may be acquired on the subjects which belong to the science, but also on account of the collateral effects and general bearings of such a study, as a discipline of the mind and an illustration of philosophical principles.

Considering the study of the mathematical sciences with reference to these latter objects, we may note two ways in which it may promote them;—by habituating the mind to strict reasoning,—and by affording an occasion of contemplating some of the most important mental processes and some of the most distinct forms of truth. Thus mathematical studies may be useful in teaching practical logic and theoretical metaphysics. We shall make a few remarks on each of these topics.

2. The study of Mathematics teaches strict reasoning—by bringing under the student's notice prominent and clear examples of trains of demonstration;—by exercising him in the habits of attentive and connected thought which are requisite in order to follow these trains;—and by familiarizing him with the peculiar and distinctive conviction which demonstration produces, and with the rigorous exclusion of all considerations which do not enter into the demonstration.

3. Logic is a system of doctrine which lays down rules for determining in what cases pretended reasonings are and are not demonstrative. And accordingly, the teaching of strict reasoning by means of the study of logic is often recommended and practised. But in order to shew the superiority of the study of mathematics for this purpose, we may consider,—that reasoning, as a practical process, must be learnt by practice, in the same manner as any other practical art, for example, riding, or fencing;—that we are not secured from committing fallacies by such a classification of fallacies as logic supplies, as a rider would not be secured from falls by a classification of them;—and that the habit of attending to our mental processes while we are reasoning, rather interferes with than assists our reasoning well, as the horseman would ride worse rather than better, if he were to fix his attention upon his muscles when he is using them.

4. To this it may be added, that the peculiar habits which enable any one to follow a *chain* of reasoning are excellently taught by mathematical study, and are hardly at all taught by logic. These habits consist in not only apprehending distinctly the demon-

stration of a single proposition when it is proved, but in retaining all the propositions thus proved, and using them in the ulterior steps of the argument with the same clear conviction, readiness, and familiarity, as if they were self-evident principles. Writers on Logic seldom give examples of reasoning in which several syllogisms follow each other; and they never give examples in which this progressive reasoning is so exemplified as to make the process familiar. Their chains generally consist only of two or three links. In Mathematics, on the contrary, every theorem is an example of such a chain; every proof consists of a series of assertions, of which each depends on the preceding, but of which the last inferences are no less evident or less easily applied than the simplest first principles. The language contains a constant succession of short and rapid references to what has been proved already; and it is justly assumed that each of these brief movements helps the reasoner forwards in a course of infallible certainty and security. Each of these hasty glances must possess the clearness of intuitive evidence, and the certainty of mature reflection; and yet must leave the reasoner's mind entirely free to turn instantly to the next point of his progress. The faculty of performing such mental processes well and readily is of great value, and is in no way fostered by the study of logic.

5. It is sometimes objected to the study of Mathematics as a discipline of reasoning, that it tends to render men insensible to all reasoning which is not mathematical, and leads them to demand, in other subjects, proofs such as the subject does not admit of, or such as are not appropriate to the matter.

To this it may be replied, that these evil results, so far as they occur, arise either from the student pursuing too exclusively one particular line of mathematical study, or from erroneous notions of the nature of demonstration.

The present volume is intended to assist, in some measure, in remedying the too exclusive pursuit of one particular line of Mathematics, by shewing that the same simplicity and evidence which are seen in the Elements of Geometry may be introduced into the treatment of another subject of a kind very different; and it is hoped that we may thus bring the subject within the reach of those who cultivate the study of Mathematics as a discipline only. The remarks now offered to the reader are intended to aid him in forming a just judgment of the analogy between mathematical and other proof; which is to be done by pointing out the true grounds of the evidence of Geometry, and by exhibiting the views which are suggested by the extension of mathematical reasoning to sciences concerned about physical facts.

6. We shall therefore now proceed to make some remarks on the nature and principles of reasoning, especially as far as they are illustrated by the mathematical sciences.

Some of the leading principles which bear upon this subject are brought into view by the consideration of the question, "What is the foundation of the certainty arising from mathematical demonstration?" and in this question it is implied that mathematical demonstration is recognised as a kind of reasoning possessing a peculiar character and evidence, which make it a definite and instructive subject of consideration.

7. Perhaps the most obvious answer to the question respecting the conclusiveness of mathematical demonstration is this;—that the certainty of such demonstration arises from its being founded upon *Axioms*; and conducted by steps, of which each might, if required, be stated as a rigorous *Syllogism*.

This answer might give rise to the further questions, What is the foundation of the conclusiveness of a Syllogism? and, What is the foundation of t certainty of an Axiom? And if we suppose the former enquiry to be left to Logic, as being the subject of that science, the latter question still remains to be considered. We may also remark upon this answer, that mathematical demonstration appears to depend upon Definitions, at least as much as upon Axioms. And thus we are led to these questions:—Whether mathematical demonstration is founded upon Definitions, or upon Axioms, or upon both? and, What is the real nature of Definitions and of Axioms?

8. The question, What is the foundation of mathematical demonstration? was discussed at considerable length by Dugald Stewart*; and the opinion at which he arrived was, that the certainty of mathemetical reasoning arises from its depending upon *definitions*. He expresses this further, by declaring that mathematical truth is hypothetical, and must be understood as asserting only, that *if* the definitions are assumed, the conclusion follows. The same opinion has, I think, prevailed widely among other modern speculators on the same subject, especially among mathematicians themselves.

* Elements of the Philosophy of the Human Mind, Vol. II.

9. In opposition to this opinion, I urge, in the first place, that no one has yet been able to construct a system of mathematical truth by means of definitions alone, to the exclusion of axioms; although attempts having this tendency have been made constantly and earnestly. It is, for instance, well known to most readers, that many mathematicians have endeavoured to get rid of Euclid's "Axioms" respecting straight lines and parallel lines; but that none of these essays has been generally considered satisfactory. If these axioms could be superseded, by definition or otherwise, it was conceived that the whole structure of Elementary Geometry would rest merely upon definitions; and it was held by those who made such essays, that this would render the science more pure, simple, and homogeneous. If these attempts had succeeded, Stewart's doctrine might have required a further consideration; but it appears strange to assert that Geometry is supported by definitions, and not by axioms, when she cannot stir four steps without resting her foot upon an axiom.

10. But let us consider further the nature of these attempts to supersede the axioms above mentioned. They have usually consisted in endeavours so to frame the definitions, that these might hold the place which the axioms hold in Euclid's reasoning. Thus the axiom, that "two straight lines cannot enclose a space," would be superfluous, if we were to take the following definition:—"A line is said to be *straight*, when two such lines cannot coincide in *two* points without coinciding *altogether*."

But when such a method of treating the subject is proposed, we are unavoidably led to ask,—whether it is allowable to lay down such a definition? It cannot

be maintained that we may propound any form of words whatever as a definition, without any consideration whether or not it suggests to the mind any intelligible or possible conception. What would be said, for instance, if we were to state the following as a definition, "A line is said to be *straight* (or any other term) when two such lines cannot coincide in *one* point without coinciding altogether?" It would inevitably be remarked, that no such lines exist; or that such a property of lines cannot hold good without other conditions than those which this definition expresses; or, more generally, that the definition does not correspond to any conception which we can call up in our minds, and therefore can be of no use in our reasonings. And thus it would appear, that a definition, to be admissible, must necessarily refer to and agree with some conception which we can distinctly frame in our thoughts.

11. This is obvious, also, by considering that the definition of a straight line could not be of any use, except we were entitled to apply it in the cases to which our geometrical propositions refer. No definition of straight lines could be employed in Geometry, unless it were in some way certain that the lines so defined are those by which angles are contained, those by which triangles are bounded, those of which parallelism may be predicated, and the like.

12. The same necessity for some general conception of such lines accompanying the definition, is implied in the terms of the definition above suggested. For what is there meant by "*such* lines?" Apparently, lines having some general character in which the property is necessarily involved. But how does it ap-

pear that lines may have such a character? And if it be self-evident that there may be such lines, this evidence is a necessary condition of this (or any equivalent) definition. And since this self-evident truth is the ground on which the course of reasoning must proceed, the simple and obvious method is, to state the property *as* a self-evident truth; that is, as an axiom. Similar remarks would apply to the other axiom above mentioned; and to any others which could be proposed on any subject of rigorous demonstration.

13. If it be conceded that such a conception accompanying the definition is necessary to justify it, we shall have made a step in our investigation of the grounds of mathematical evidence. But such an admission does not appear to be commonly contemplated by those who maintain that the conclusiveness of mathematical proof results from its depending on definitions. They generally appear to understand their tenet as if it implied *arbitrary* definitions. And something like this seems to be held by Stewart, when he says that mathematical truths are true *hypothetically*. For we understand by an hypothesis a supposition, not only which we may make, but may abstain from making, or may replace by a different supposition.

14. That the fundamental conceptions of Geometry are not arbitrary definitions, or selected hypotheses, will, I think, be clear to any one who reasons geometrically at all. It is impossible to follow the steps of any single proposition of Geometry without conceiving a straight line and its properties, whether or not such a line be defined, and whether or not its properties be stated. That a straight line should be distinguished from all other lines, and that the axiom

respecting it should be seen to be true, are circumstances indispensable to any clear thought on the subject of lines. Nor would it be possible to frame any coherent scheme of Geometry in which straight lines should be excluded, or their properties changed. Any one who should make the attempt, would betray, in his first propositions, to all men who can reason geometrically, a reference to straight lines.

15. If, therefore, we say that Geometry depends on definitions, we must add, that they are *necessary*, not arbitrary definitions,—such definitions as we must have in our minds, so far as we have elements of reasoning at all. And the elementary hypotheses of Geometry, if they are to be so termed, are not hypotheses which are requisite to enable us to reach this or that conclusion; but hypotheses which are requisite for *any* exercise of our thoughts on such subjects.

16. Before I notice the bearing of this remark on the question of the necessity of axioms, I may observe that Stewart's disposition to consider definitions, and not axioms, as the true foundation of Geometry, appears to have resulted, in part, from an arbitrary selection of certain axioms, as specimens of all. He takes, as his examples, the axioms, "that if equals be added to equals the wholes are equal," that "the whole is greater than its part;" and the like. If he had, instead of these, considered the more properly geometrical axioms,—such as those which I have mentioned; "that two straight lines cannot enclose a space;" or any of the axioms which have been made the basis of the doctrine of parallels; for instance, Playfair's axiom, "that two straight lines which intersect each other cannot both of them be parallel to a third straight

line;" it would have been impossible for him to have considered axioms as holding a different place from definitions in geometrical reasoning. For the properties of triangles are proved from the axiom respecting straight lines, as distinctly and directly, as the properties of angles are proved from the definition of a right angle. Of the many attempts made to prove the doctrine of parallels, almost all professedly, all really, assume some axiom or axioms which are the basis of the

17. It is therefore very surprising that Stewart should so exclusively have fixed his attention upon the more general axioms, as to assert, following Locke, "that from [mathematical] Axioms it is not possible for human ingenuity to draw a single inference*;" and even to make this the ground of a contrast between geometrical Axioms and Definitions. The slightest examination of any treatise of Geometry might have shewn him that there is no sense in which this can be asserted of Axioms, in which it is not equally true of Definitions; or rather, that while Euclid's Definition of a straight line leads to no truth whatever, his Axiom respecting straight lines is the foundation of the whole of Geometry; and that, though we can draw some inferences from the Definition of parallel straight lines, we strive in vain to complete the geometrical doctrine of such lines, without assuming some Axiom which enables us to prove the converse of our first propositions. Thus, that which Stewart proposes as the distinctive character of Axioms, fails altogether; and with it, as I conceive, the whole of his doctrine respecting mathematical evidence.

* Elements of the Philosophy of the Human Mind, Vol. II. p. 38.

18. That Geometry (and other sciences when treated in a method equally rigorous) depends upon axioms as well as definitions, is supposed by the form in which it is commonly presented. And after what we have said, we shall assume this form to be a just representation of the real foundations of such sciences, till we can find a tenable distinction between axioms and definitions, in their nature, and in their use; and till we have before us a satisfactory system of Geometry without Axioms. And this system, we may remark, ought to include the Higher as well as the Elementary Geometry, before it can be held to prove that axioms are needless; for it will hardly be maintained, that the properties of circles depend upon definitions and hypotheses only, while those of ellipses require some additional foundation: or that the comparison of curve lines requires axioms, while the relations of straight lines are independent of such principles.

19. Having then, I trust, cleared away the assertion, that mathematical reasoning rests ultimately upon definitions only, and that this is the ground of its peculiar cogency, I have to examine the real evidence of the truth of such axioms as are employed in the exact Mathematical Sciences. And we are, I think, already brought within view of the answer to this question. For if the definitions of Mathematics are not arbitrary, but necessary, and must, in order to be applicable in reasoning, be accompanied by a conception of the mind through which this necessity is seen; it is clear that this apprehension of the necessity of the properties which we contemplate, is really the ground of our reasonings and the source of their irresistible evidence. And where we clearly apprehend such necessary rela-

tions, it can make no difference whatever in the nature of our reasoning, whether we express them by means of definitions or of axioms. We define a straight line vaguely;—that it is that line which lies *evenly* between two points: but we forthwith remedy this vagueness, by the axiom respecting straight lines: and thus we express our conception of a straight line, so far as is necessary for reasoning upon it. We might, in like manner, begin by defining a right angle to be the angle made by a line which stands evenly between the two portions of another line; and we might add an axiom, that all right angles are equal. Instead of this, we define a right angle to be that which a line makes with another when the two angles on the two sides of it are equal. But in all these cases, we express our conception of a necessary relation of lines; and whether this be done in the form of definitions or axioms, is a matter of no importance.

20. But it may be asked, If it be thus unimportant whether we state our fundamental principles as axioms or definitions, why not reduce them all to definitions, and thus give to our system that aspect of independence which many would admire, and with which none need be displeased? And to this we answer, that if such a mode of treating the subject were attempted, our definitions would be so complex, and so obviously dependent on something not expressed, that they would be admired by none. We should have to put into each definition, as conditions, all the axioms which refer to the things defined. For instance, who would think it a gain to escape the difficulties of the doctrine of parallels by such a definition as this: "Parallel straight lines are those

which being produced indefinitely both ways do not meet; and which are such that if a straight line intersects one of them it must somewhere meet the other?" And in other cases, the accumulation of necessary properties would be still more cumbersome and more manifestly heterogeneous.

21. The reason of this difficulty is, that our fundamental conceptions of lines and other relations of space, are capable of being contemplated under several various aspects, and more than one of these aspects are needed in our reasonings. We may take one such aspect of the conception for a definition; and then we must introduce the others by means of axioms. We may define parallels by their not meeting; but we must have some positive property, besides this negative one, in order to complete our reasonings respecting such lines. We have, in fact, our choice of several such self-evident properties, any of which we may employ for our purpose, as geometers well know; but with our naked definition, as they also know, we cannot proceed to the end. And in other cases, in like manner, our fundamental conception gives rise to various elementary truths, the connexion of which is the basis of our reasonings: but this connexion resides in our thoughts, and cannot be made to follow, as a logical result, from any assumed form of words, presented as a definition.

22. If it be further demanded, What is the nature of this bond in our thoughts by which various properties of lines are connected? perhaps the simplest answer is to say, that it resides in *the idea of space.* We cannot conceive things in space without being led to consider them as determined and related in some way

or other to straight lines, right angles, and the like; and we cannot contemplate these determinations and relations distinctly, without assuming those properties of straight lines, of right angles, and of the rest, which are the basis of our Geometry. We cannot conceive or perceive objects at all, except as existing in space; we cannot contemplate them geometrically, without conceiving them in space which is subjected to geometrical conditions; and this mode of contemplation is, by language, analysed into definitions, axioms, or both.

23. The truths thus seen and known, may be said to be known by *intuition*. In English writers this term has, of late, been vaguely used, to express all convictions which are arrived at without conscious reasoning, whether referring to relations among our primary perceptions, or to conceptions of the most derivative and complex nature. But if we were allowed to restrict the use of this term, we might conveniently confine it to those cases in which we necessarily apprehend relations of things truly, as soon as we conceive the objects distinctly. In this sense *axioms* may be said to be *known by intuition*; but this phraseology is not essential to our purpose.

24. It appears, then, that the evidence of the axioms of Geometry depends upon a distinct possession of the idea of space. These axioms are stated in the beginning of our Treatises, not as something which the reader is to learn, but as something which he already knows. No proof is offered of them; for they are the beginning not the end of demonstrations. The student's clear apprehension of the truth of these is a condition of the possibility of his pursuing the reason-

ings on which he is invited to enter*. Without this mental capacity, and the power of referring to it, in the reader, the writer's assertions and arguments are empty and unmeaning words; but then, this capacity and power are what all rational creatures alike possess, though habit may have developed it in very various degrees in different persons.

25. It has been common in the school of Metaphysicians of which I have spoken, to describe some of the elementary convictions of our minds as *fundamental laws* of belief; and it appears to have been considered that this might be taken as a final and sufficient account of such convictions. I do not know whether any persons would be tempted to apply this formula, as a solution of our question respecting the nature of axioms. If this were proposed, I should observe, that this form of expression seems to me, in such a case, highly unsatisfactory. For *laws* require and enjoin a conjunction of things which can be contemplated separately, and which would be disjoined if the law did not exist. It is a law of nature that terrestrial bodies, when free,

* In this statement respecting the nature of Axioms, I find myself agreeing with the acute author of "Sematology." See the "Sequel to Sematology," p. 103. "An Axiom does not account for an intellection; it does but describe the requisite competency for it." It appears to me that this view is not familiar among English metaphysicians. I may here quote what I said at a former period, "However we may *define* force, it is necessary, in order to understand the elementary reasonings of this portion of science, that we should *conceive* it distinctly. Do we wish for a test of the distinctness of our conceptions? The test is, our being able to see the necessary truth of the Axioms on which our reasonings rest ..These principles (the Axioms of Statics) are all perfectly evident as soon as we have formed the general conception of pressure; but without that act of thought, they can have no evidence whatever given them by any form of words, or reference to other truths;—by definitions, or by illustrations from other kinds of quantity."—*Thoughts on the Study of Mathematics*, p. 25.

fall downwards; for we can easily conceive such bodies divested of such a property. But we cannot say, in the same sense, that the impossibility of two straight lines inclosing a space arises from a law; for if they are straight lines, they need no law to compel this result. We cannot conceive straight lines exempt from such a law. To speak of this property as imposed by a law, is to convey an inadequate and erroneous notion of the close necessity, inviolable even in thought, by which the truth clings to the conception of the lines.

26. This expression, of "laws of belief," appears to have found favour, on this account among others, that it recognises a kind of analogy between the grounds of our reasoning on very abstract subjects, and the principles to which we have recourse in other cases when we manifestly derive our fundamental truths from facts, and when it is supposed to be the ultimate and satisfactory account of them to say, that they are laws of nature learnt by observation. But such an analogy can hardly be considered as a real recommendation by the metaphysician; since it consists in taking a case in which our knowledge is obviously imperfect and its grounds obscure, and in erecting this case into an authority which shall direct the process and control the enquiry of a much more profound and penetrating kind of speculation. It cannot be doubted that we are likely to see the true grounds and evidence of our doctrines much more clearly in the case of Geometry and other rigorous systems of reasoning, than in collections of mere empirical knowledge, or of what is supposed to be such. It is both an unphilosophical and an indolent proceeding, to take the latter cases as a standard for the former.

27. I shall therefore consider it as established, that in Geometry our reasoning depends upon axioms as well as definitions,—that the evidence of the truth of the axioms and of the propriety of the definitions resides in the idea of space,—and that the distinct possession of this idea, and the consequent apprehension of the truth of the axioms which are its various aspects, is supposed in the student who is to pursue the path of geometrical reasoning. This being understood, I have little further to observe on the subject of Geometry. I will only remark,—that all the conclusions which occur in the science follow purely from those first principles of which we have spoken;—that each proposition is rigorously proved from those which have been proved previously from such principles;—that this process of successive proof is termed *Deduction*;—and that the rules which secure the rigorous conclusiveness of each step are the rules of *Logic*, which I need not here dwell upon.

28. But I now proceed to consider some other questions to which our examination of the evidence of Geometry was intended to be preparatory;—How far do the statements hitherto made apply to other sciences? for instance, to such sciences as are treated of in the present volume, Mechanics and Hydrostatics. To this I reply, that some such sciences at least, as for example the science of Statics, appear to me to rest on foundations exactly similar to Geometry:—that is to say, that they depend upon axioms,—self-evident principles, not derived in any immediate manner from experiment, but involved in the very nature of the conceptions which we must possess, in order to reason upon such subjects at all. The proof of this doctrine must consist of several steps, which I shall take in order.

29. In the first place, I say that the axioms of Statics are *self-evidently true.* In the beginning of the preceding Treatise I have stated these barely as axioms, without addition or explanation, as the axioms of Geometry are stated in treatises on that subject. And such is the proper and orderly mode of exhibiting axioms; for, as has been said, they are to be understood as an expression of the condition of conception of the student. They are not to be learnt from without, but from within. They necessarily and immediately flow from the distinct possession of that idea, which if the student do not possess distinctly, all conclusive reasoning on the subject under notice is impossible. It is not the business of the deductive reasoner to communicate the apprehension of these truths, but to deduce others from them.

30. But though it may not be the author's business to elucidate the truth of the axioms as a deductive reasoner, it may still be desirable that he should do so as a philosophical teacher; and though it may not be possible to add anything to their evidence in the mind of him who possesses distinctly the idea from which they flow, it may be in our power to assist the beginner in obtaining distinct possession of this idea and unfolding it into its consequences. Accordingly I have made some Remarks of this kind, tending to illustrate the self-evident nature of the "Axioms" of Statics and of Hydrostatics; and have inserted them in Book I. and Book II. respectively.

31. Some of the Axioms which are stated in Book III, on the Laws of Motion, give occasion to remarks similar to those already made. Thus Axiom 4,

which asserts that if particles move in such a manner as always to preserve the same relative distances and positions, their motions will not be altered by supposing them rigidly connected, is evident by the same considerations as the Axioms concerning flexible and fluid bodies, already noticed in Book II. For the forces of rigidity are forces which would prevent a change of the distances and relative positions of the particles if there were a tendency to any such change; and if there be no such tendency, it makes no difference whether the potential resistance to it be present or absent.

32. The 5th Axiom of Book III. which asserts that forces producing parallel and equal velocities at the same time, may be conceived to be added; and the 6th Axiom, which asserts that in systems in motion the action and re-action are equal and opposite, are applications of what is stated in the second sentence of this third Book;—that the Definitions and Axioms of Statics are adopted and assumed in the case of bodies in motion. In the third Book, as in the first, forces are conceived as capable of addition, and matter is conceived as that which can resist force, and transmit it unaltered.

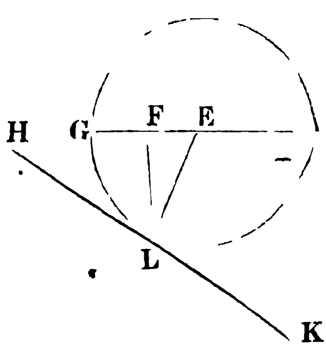
The 3d, 8th, and 9th Axioms of Book III, like the 7th of Book II, are introduced to avoid the reasoning which depends on Limits.

33. In the case of Mechanics, as in the case of Geometry, the distinctness of the idea is necessary to a full apprehension of the truth of the axioms; and in the case of mechanical notions it is far more common than in Geometry, that the axioms are imperfectly comprehended, in consequence of the want

of distinctness and exactness in men's ideas. Indeed this indistinctness of mechanical notions has not only prevailed in many individuals at all periods, but we can point out whole centuries, in which it has been, so far as we can trace, universal. And the consequence of this was, that the science of Statics, after being once established upon clear and sound principles, again fell into confusion, and was not understood as an exact science for two thousand years, from the time of Archimedes to that of Galileo and Stevinus.

34. In order to illustrate this indistinctness of mechanical ideas, I shall take from an ancient Greek writer an attempt to solve a mechanical problem; namely, the Problem of the Inclined Plane. The following is the mode in which Pappus professes* to answer this question:—"To find the force which will support a given weight *A* upon an inclined plane."

Let *HK* be the plane; let the weight *A* be formed into a sphere: let this sphere be placed in contact with the plane *HK*, touching it in the point *L*, and let *E* be its center. Let *EG* be a horizontal radius, and *LF* a vertical line which meets it. Take a weight *B* which is to *A* as *EF* to *FG*. Then if *A* and *B* be suspended at *E* and *G* to the lever *EFG*



of which the center of motion is *F*, they will balance; being supported, as it were, by the fulcrum *LF*. And the sphere, which is equal to the weight *A*, may

* Pappus, B. VIII. Prop. ix. I purposely omit the confusion produced by this author's mode of treating the question, in which he inquires the force which will *draw* a body *up* the inclined plane.

be supposed to be collected at its center. If therefore B act at G, the weight A will be supported.

It may be observed that in this attempt, the confusion of ideas is such, that the author assumes a weight which acts at G, perpendicularly on the lever EFG, and which is therefore a vertical force, as identical with a force which acts at G, to support the body in the inclined plane, and which is parallel to the plane.

35. When this kind of confusion was remedied, and when men again acquired distinct notions of pressure, and of the transmission of pressure from one point to another, the science of Statics was formed by Stevinus, Galileo, and their successors *.

The fundamental idea of Mechanics being thus acquired, and the requisite consequences of them stated in axioms, our reasonings proceed by the same rigorous line of demonstration, and under the same logical rules as the reasonings of Geometry; and we have a science of Statics which is, like Geometry, an exact deductive science.

SECT. II. *On the Logic of Induction.*

36. There are other portions of Mechanics which require to be considered in another manner; for in these there occur principles which are derived directly and professedly from experiment and observation. The derivation of principles by reasoning from facts is performed by a process which is termed *Induction*, which is very different from the process of Deduction already noticed, and of which we shall attempt to point out the character and method.

* See History of the Inductive Sciences, B. VI. chap. I. sect. 2, *On the Revival of the Scientific Idea of Pressure.*

It has been usual to say of any general truths, established by the consideration and comparison of several facts, that they are obtained by *Induction*; but the distinctive character of this process has not been well pointed out, nor have any rules been laid down which may prescribe the form and ensure the validity of the process, as has been done for Deductive reasoning by common Logic. The *Logic of Induction* has not yet been constructed; a few remarks on this subject are all that can be offered here.

37. The Inductive Propositions, to which we shall here principally refer as examples of their class, are those elementary principles which occur in considering the motion of bodies, and of which some are called the Laws of Motion *. They are such as these;—a body not acted on by any force will move on for ever uniformly in a straight line;—gravity is a uniform force;—if a body in motion be acted upon by any force, the effect of the force will be compounded with the previous motion;—when a body communicates motion to another directly, the momentum lost by the first body is equal to the momentum gained by the second. And I remark, in the first place, that in collecting such propositions from facts, there occurs a step corresponding to the term "Induction," (ἐπαγωγή, *inductio*). Some notion is *superinduced* upon the observed facts. In each inductive process, there is some general idea introduced, which is given, not by the phenomena, but by the mind. The conclusion is not contained in the premises, but includes them by the introduction of a new generality. In order to obtain our inference, we travel beyond the cases we

* Inductive Propositions in this work are, Book II. Propositions 25, 26 32, 36, 37: Book III. Prop. 2, 3, 8, 13.

have before us; we consider them as exemplifications of, or deviations from, some ideal case in which the relations are complete and intelligible. We take a standard, and measure the facts by it; and this standard is created by us, not offered by Nature. Thus we assert, that a body left to itself will move on with unaltered velocity, not because our senses ever disclosed to us a body doing this, but because (taking this as our ideal case) we find that all actual cases are intelligible and explicable by means of the notion of forces which cause change of motion, and which are exerted by surrounding bodies. In like manner, we see bodies striking each other, and thus moving, accelerating, retarding, and stopping each other; but in all this, we do not, by our senses, perceive that abstract quantity, *momentum*, which is always lost by one as it is gained by another. This momentum is a creation of the mind, brought in among the facts, in order to convert their apparent confusion into order, their seeming chance into certainty, their perplexing variety into simplicity. This the idea of *momentum gained and lost* does; and, in like manner in any other case in which inductive truths are established, some idea is introduced, as the means of passing from the facts to the truth.

38. The process of mind of which we here speak can only be described by suggestion and comparison. One of the most common of such comparisons, especially since the time of Bacon, is that which speaks of induction as the *interpretation* of facts. Such an expression is appropriate; and it may easily be seen that it includes the circumstance which we are now noticing;—the superinduction of an idea upon the facts by the interpreting mind. For when we read a

page, we have before our eyes only black and white, form and colour; but by an act of the mind, we transform these perceptions into thought and emotion. The letters are nothing of themselves; they contain no truth, if the mind does not contribute its share: for instance, if we do not know the language in which the words are written. And if we are imperfectly acquainted with the language, we become very clearly aware how much a certain activity of the mind is requisite in order to convert the words into propositions, by the extreme effort which the business of interpretation requires. Induction, then, may be conveniently described as the *interpretation of phenomena.*

39. But I observe further, that in thus inferring truths from facts, it is not only necessary that the mind should contribute to the task its own idea, but, in order that the propositions thus obtained may have any exact import and scientific value, it is requisite that the idea be perfectly *distinct* and precise. If it be possible to obtain some vague apprehension of truths, while the ideas in which they are expressed remain indistinct and ill-defined, such knowledge cannot be available for the purposes we here contemplate. In order to construct a science, all our fundamental ideas must be distinct; and among them, those which Induction introduces.

40. This necessity for distinctness in the ideas which we employ in Induction, makes it proper to *define*, in a precise and exact manner, each idea when it is thus brought forwards. Thus, in establishing the propositions which we have stated as our examples in these cases, we have to define *force* in general; *uni-*

form force; compounding of motions; *momentum.* The construction of these definitions is an essential part of the process of Induction, no less than the assertion of the inductive truth itself.

41. But in order to justify and establish the inference which we make, the ideas which we introduce most not only be distinct, but also *appropriate.* They must be exactly and closely applicable to the facts; so that when the idea is in our possession, and the facts under our notice, we perceive that the former includes and takes up the latter. The idea is only a more precise mode of apprehending the facts, and it is empty and unmeaning if it be anything else; but if it be thus applicable, the proposition which is asserted by means of it is true, precisely because the facts *are* facts. When we have defined *force* to be *the cause of change of motion,* we see that, as we remove *external* forces, we do, in actual experiments, remove all the change of motion; and therefore the proposition that there is in bodies no *internal* cause of change of motion, is true. When we have defined *momentum* to be the *product of the velocity and quantity of matter,* we see that in the actions of bodies, the *effect* increases as the *momentum* increases; and by measurement, we find that the effect may consistently be *measured* by the momentum. The ideas here employed are not only distinct in the mind, but applicable in the world; they are the elements, not only of relations of thought, but of laws of nature.

42. Thus an inductive inference requires an idea from within, facts from without, and a coincidence of the two. The idea must be distinct, otherwise we obtain no scientific truth; it must be appropriate, other-

wise the facts cannot be steadily contemplated by means of it; and when they are so contemplated, the Inductive Proposition must be seen to be verified by the evidence of sense.

It appears from what has been said, that in establishing a proposition by Induction, the *definition* of the *idea* and the *assertion* of the *truth*, are not only both requisite, but they are correlative. Each of the two steps contains the verification and justification of the other. The proposition derives its meaning from the definition; the definition derives its reality from the proposition. If they are separated, the definition is arbitrary or empty, the proposition is vague or verbal.

43. Hence we gather, that in the Inductive Sciences, our Definitions and our Elementary Inductive Truths ought to be introduced together. There is no value or meaning in definitions, except with reference to the truths which they are to express. Discussions about the definitions of any science, taken separately, cannot therefore be profitable, if the discussion do not refer, tacitly or expressly, to the fundamental truths of the science; and in all such discussions, it should be stated what are taken as the fundamental truths. With such a reference to Elementary Inductive Truths clearly understood, the discussion of Definitions may be the best method of arriving at that clearness of thought, and that arrangement of facts, which Induction requires.

I will now note some of the differences which exist between Inductive and Deductive Reasoning, in the modes in which they are presented.

44. One leading difference in these two kinds of reasoning is, that in Deduction we infer particular

from general truths; in Induction, on the contrary, we infer general from particular. Deductive proof consists of many steps, in each of which we apply known general propositions in particular cases;—"all triangles have their angles equal to two right angles, therefore this triangle has; therefore, &c." In Induction, on the other hand, we have a single step in which we pass from many particular Propositions to one general proposition; "This stone falls downwards; so do those others;—all stones fall downwards." And the former inference flows necessarily from the relation of general and particular; but the latter, as we have seen, derives its power of convincing from the introduction of a new idea, which is distinct and appropriate, and which supplies that generality which the particulars cannot themselves offer.

45. I observe also that this difference of process in inductive and deductive proofs, may be most properly marked by a difference in the form in which they are stated. In Deduction, the *Definition* stands at the beginning of the proposition; in Induction, it may most suitably stand at or near the end. Thus the definition of a uniform force is introduced in the course of the psoposition that gravity is a uniform force. And this arrangement represents truly the real order of proof; for, historically speaking, it was taken for granted that gravity was a uniform force; but the question remained, what was the right definition of a uniform force. And in the establishment of other inductive principles, in like manner, definitions cannot be laid down for any useful purpose, till we know the propositions in which they are to be used. They may therefore properly come each at the conclusion of its corresponding proposition.

46. The ideas and definitions to which we are thus led by our inductive process, may bring with them Axioms. Such Axioms may be self-evident as soon as the inductive idea has been distinctly apprehended, in the same manner as was explained respecting the fundamental ideas of Geometry and Statics. And thus *Axioms*, as well as Definitions, may come at the end of our Inductive Propositions; and they thus assume their proper place at the beginning of the deductive propositions which follow them, and are proved from them. Thus, in Book III., Axioms 8 and 9, come after the definition of Accelerating Force, and stand between Props. 13 and 14.

47. Another peculiarity in inductive reasoning may be noticed. In a deductive demonstration, the reference is always to what has been already proved; in establishing an Inductive Principle, it is most convenient that the reference should be to subsequent propositions. For the proof of the Inductive Principle consists in this;—that the principle being adopted, consequences follow which agree with fact; but the demonstration of these consequences may require many steps, and several special propositions. Thus the Inductive Principle, that gravity is a uniform force, is established by shewing that the law of descent, which falling bodies follow in fact, is explained by means of this principle; namely, the law that the space is as the square of the time from the beginning of the motion. But the proof of such a property, from the definition of a uniform force, requires many steps, as may be seen in the preceding Treatise, Book III. Prop. 5: and this proof must be referred to, along with several others, in order to establish the truth, that gravity is a uniform force.

48. It may be suggested, that, this being the case, the propositions might be transposed, so that the inductive proof might come after those propositions to which it refers. But if this were done, all the propositions which depend upon the laws of motion must be proved hypothetically only. For instance, we must say, "*If*, in the communication of motion, the momentum lost and gained be equal, the velocity acquired by a body falling down an inclined plane, will be equal to that acquired by falling down the height." This would be inconvenient, and even if it were done, that completeness in the line of demonstration which is the object of the change, could not be obtained; for the transition from the particular cases to the general truth, which must occur in the Inductive Proposition, could not be in any way justified according to rules of Deductive Logic.

I have, therefore, in the preceding pages, placed the Inductive Principle first in each line of reasoning, and have ranged after it the Deductions from it, which justify and establish it, as their first office, but which are more important as its consequences and applications, after it is supposed to be established.

49. I have used one common *formula* in presenting the proof of each of the Inductive Principles which I have introduced; namely, after stating or exemplifying the facts which the induction includes, I have added "These results can be clearly explained and rigorously deduced by introducing the *Idea* or the *Definition*," which belongs to each case, "and the *Principle*," which expresses the inductive truth. I do not mean to assert that this formula is the only right one, or even the best; but it appears to me to

bring under notice the main circumstances which render an induction systematic and valid.

50. It may be observed, however, that this formula does not express the full cogency of the proof. It declares only that the results *can* be clearly explained and rigorously deduced by the employment of a certain definition and a certain proposition. But in order to make the conclusion demonstrative, we ought to be able to declare that the results can be clearly explained and rigorously deduced *only* by the definition and proposition which we adopt. And, in reality, the mathematician's conviction of the truth of the Laws of Motion does depend upon his seeing that they (or laws equivalent to them) afford the *only* means of clearly expressing and deducing the actual facts. But this conviction, that no other law than those proposed can account for the known facts, finds its place in the mind gradually, as the contemplation of the consequences of the law and the various relations of the facts becomes steady and familiar. I have therefore not thought it proper to require such a conviction along with the first assent to the inductive truths which I have here stated.

51. The propositions established by Induction are termed *Principles*, because they are the starting points of trains of deductive reasoning. In the system of deduction, they occupy the same place as axioms; and accordingly they are termed so by Newton—"Axiomata sive leges motus." Stewart objects strongly to this expression*: and it would be difficult to justify it; although to draw the line be-

* Elem. Phil. Human Mind. Vol. II. p. 44.

tween Axioms and inductive principles may be a harder task than at first appears.

52. But from the consideration that our Inductive Propositions are the principles or beginnings of our deductive reasoning, and so far at least stand in the place of axioms, we may gather *this* lesson,—that they are not to be multiplied without necessity. For instance, if in a treatise on Hydrostatics, we should state as two separate propositions, that "air has weight;" and that "the mercury in the barometer ~~is sustained~~ by the weight of the air;" and should prove both the one and the other by reference to experiment; we should offend against the maxims of Logic. These propositions are connected; the latter may be demonstrated deductively from the former; the former may be inferred inductively from the facts which prove the latter. One of these two courses ought to be adopted; we ought not to have two ends of our reasoning upwards, or two beginnings of our reasoning downwards.

53. I shall not now extend these Remarks further. They may appear to many barren and unprofitable speculations; but those who are familiar with such subjects, will perhaps find in them something which, if well founded, is not without some novelty for the English reader. Such will, I think, be the case, if I have satisfied him,—that mathematical truth depends on Axioms as well as Definitions,—that the evidence of geometrical Axioms is to be found only in the distinct possession of the Idea of Space,—that other branches of mathematics also depend on Axioms,—and that the evidence of these Axioms is to be sought in some appropriate Idea;—that the evidence of the Axioms of Statics, for instance, resides in the Ideas of Force and

Matter;—that in the process of Induction the mind must supply an Idea in addition to the Facts apprehended by the senses;—that in each such process we must introduce one or more Definitions, as well as a Proposition;—that the Definition and the Proposition are correlative, neither being useful or valid without the other;—and that the Formula of inductive reasoning must be in many respects the reverse of the common logical formulæ of deduction.

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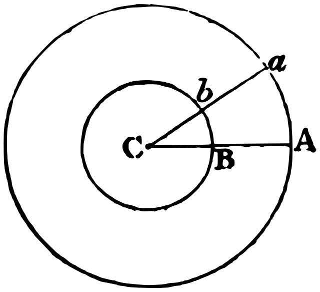
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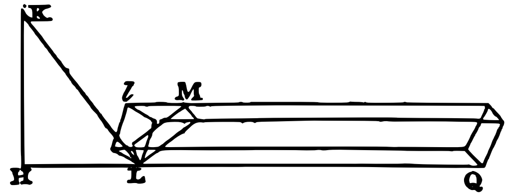
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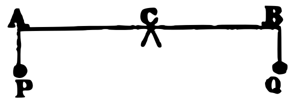
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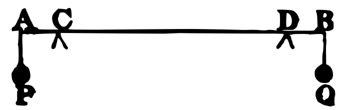
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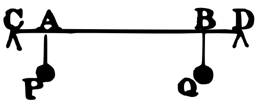
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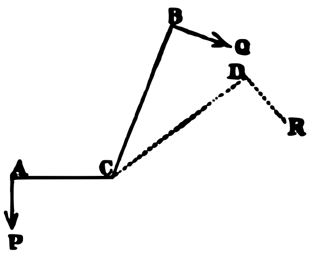
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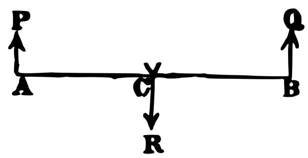
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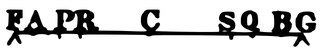
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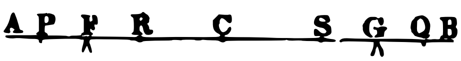
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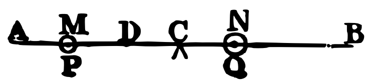
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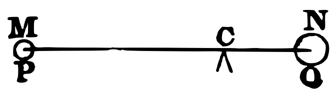
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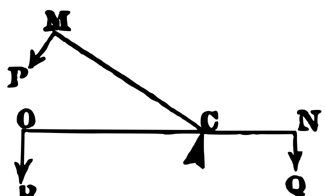
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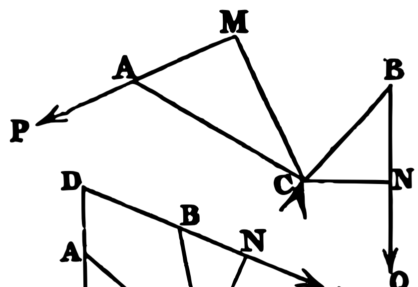
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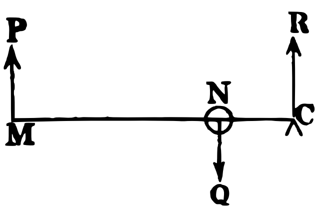
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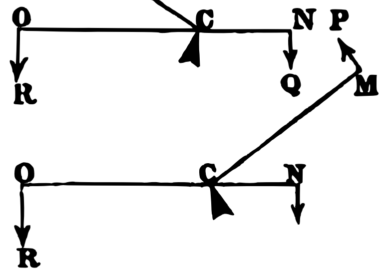
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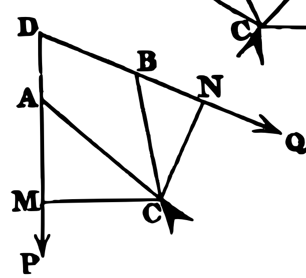
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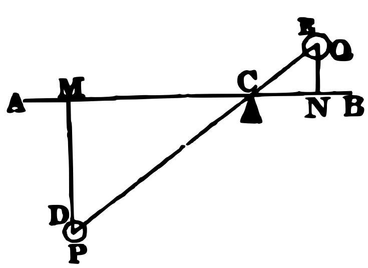
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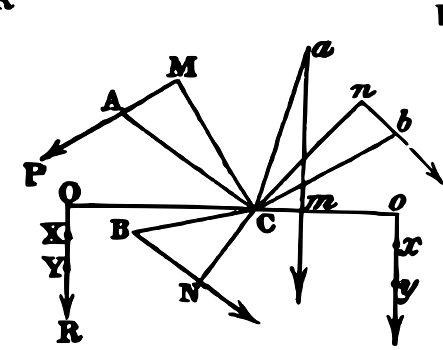
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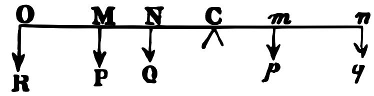
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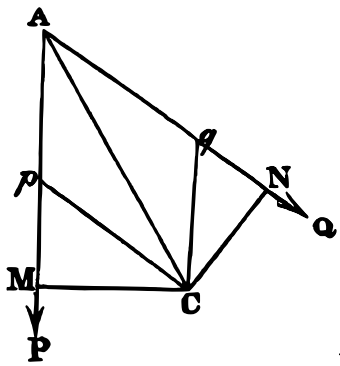
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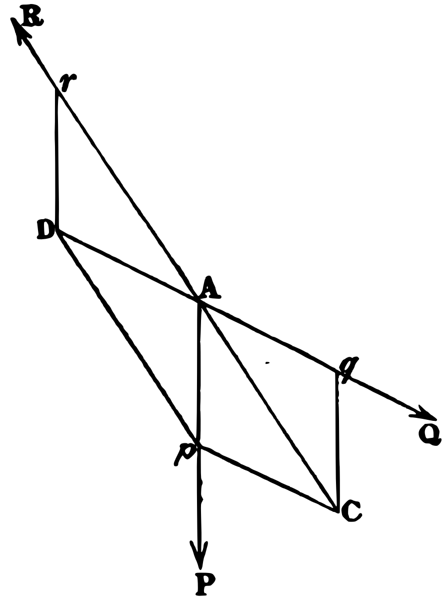
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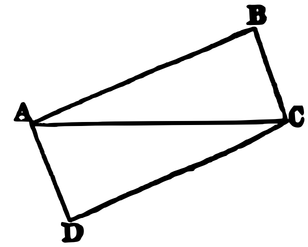
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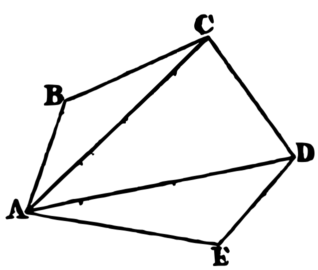
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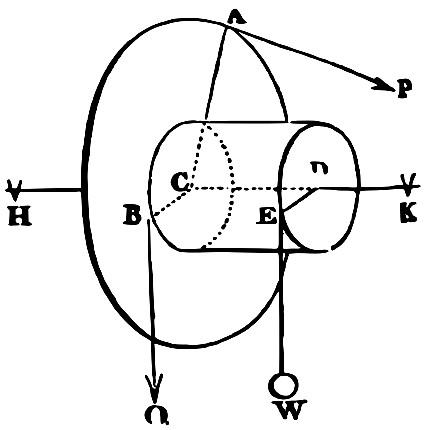
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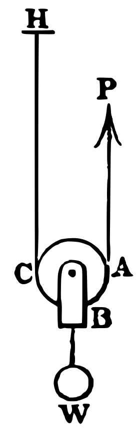
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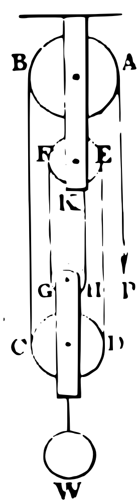
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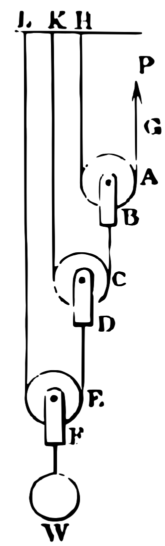
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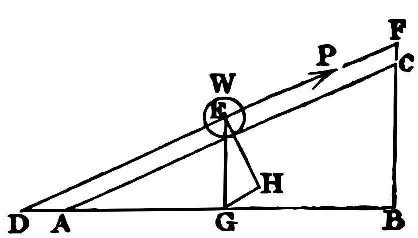
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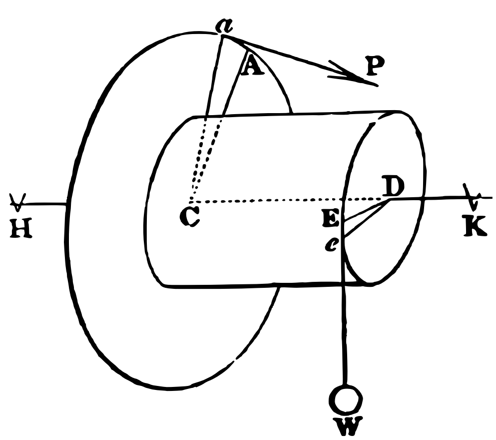
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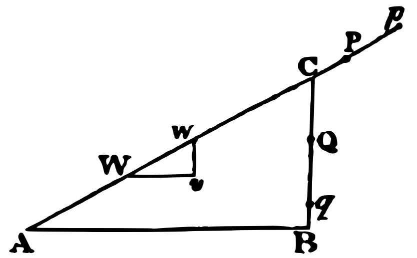

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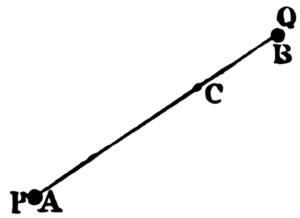

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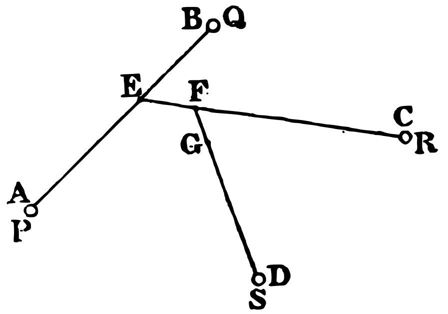

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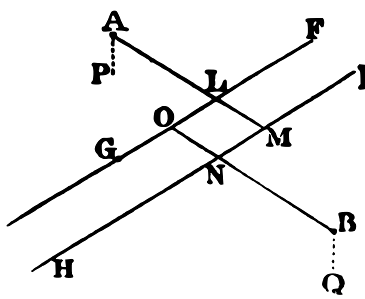

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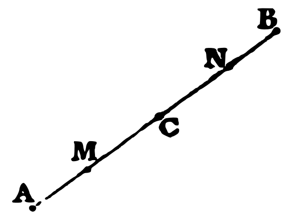

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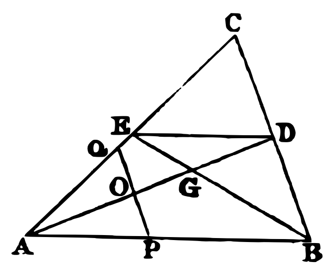

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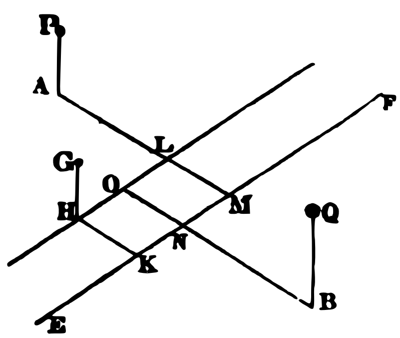

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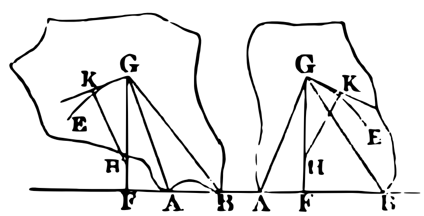

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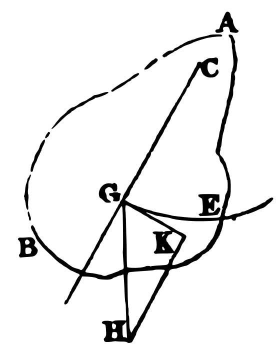

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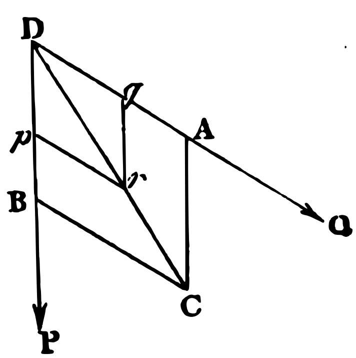

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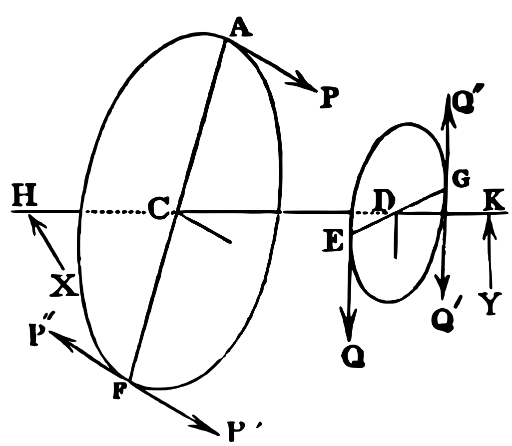

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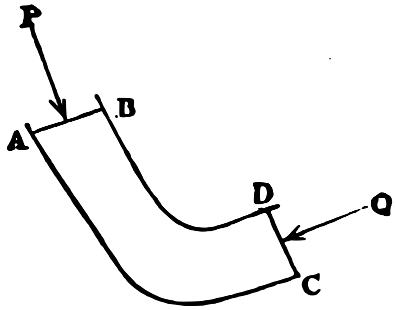
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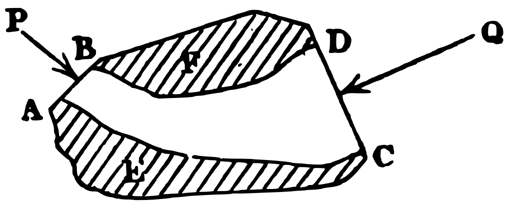
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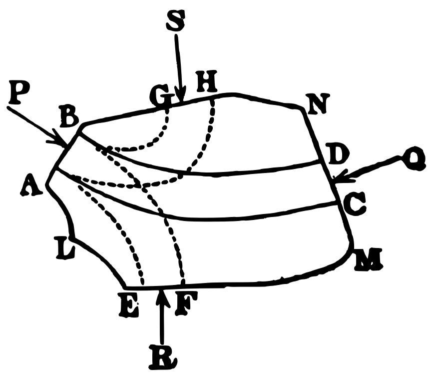
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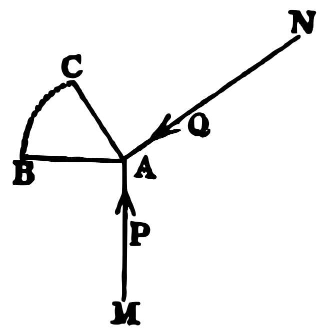
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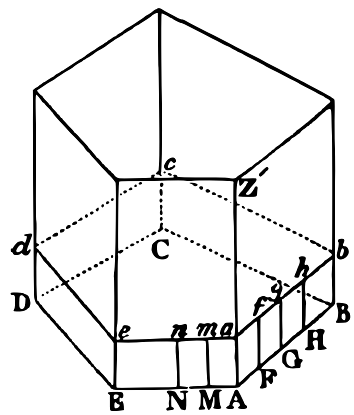
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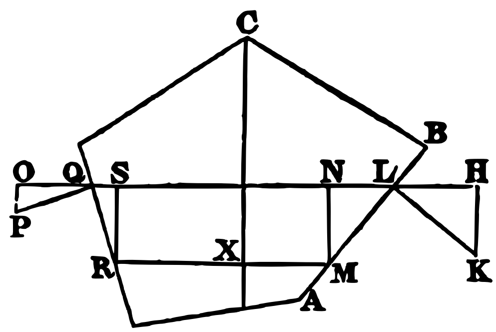
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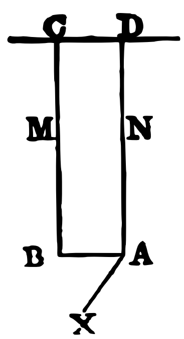
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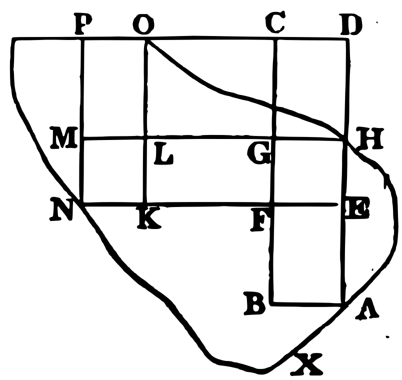
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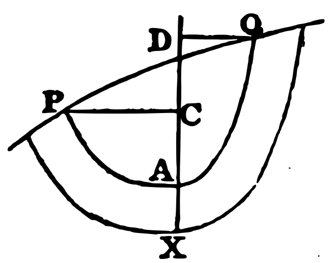
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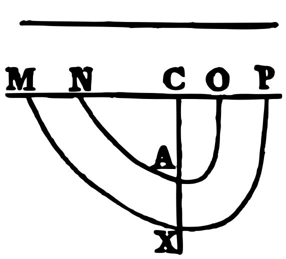
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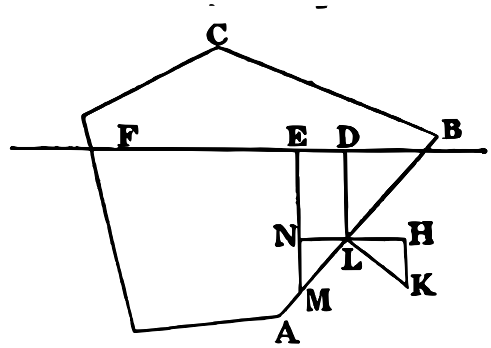
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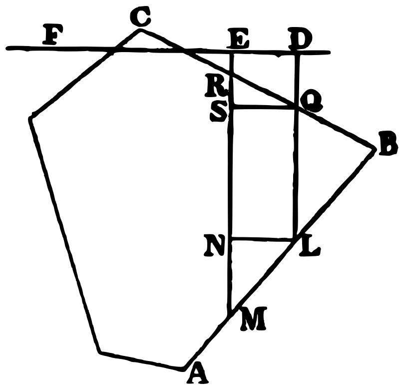
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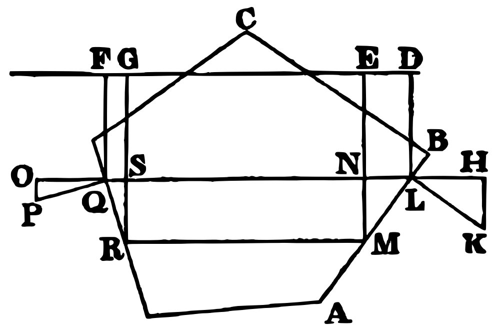
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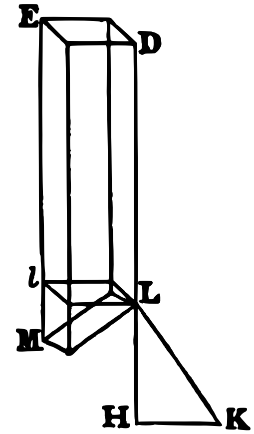
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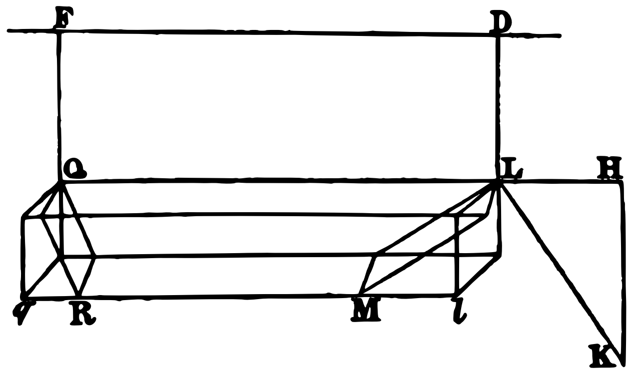
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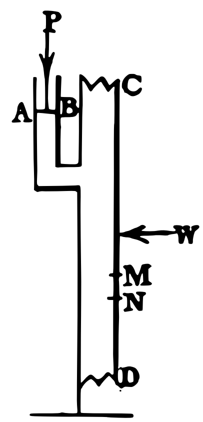
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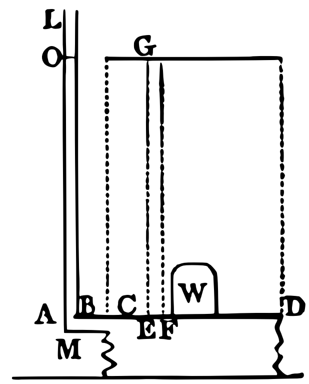
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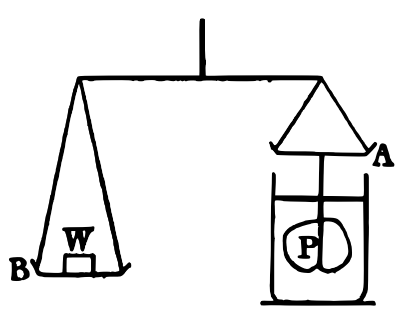
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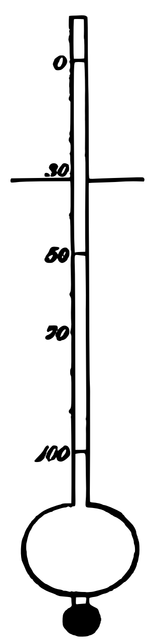
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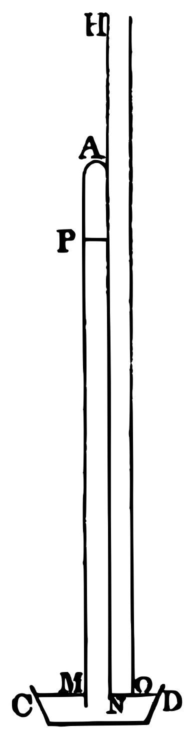
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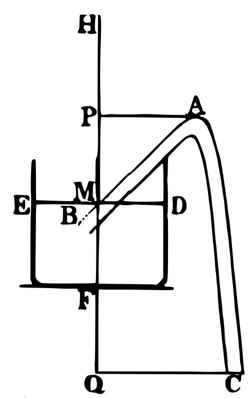
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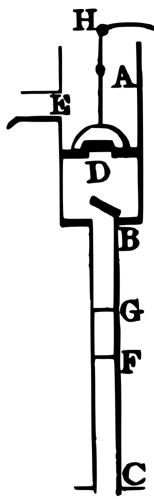
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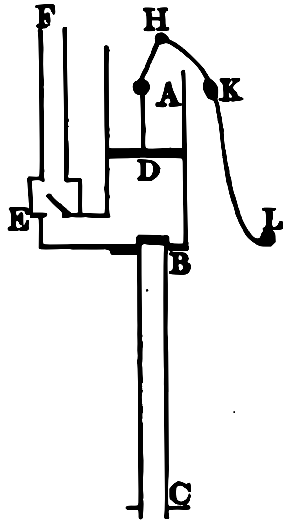
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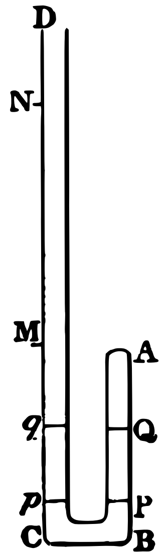
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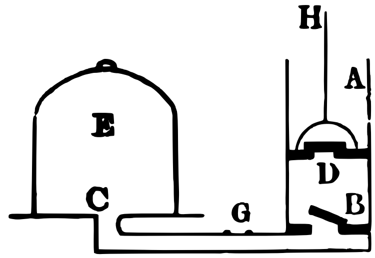
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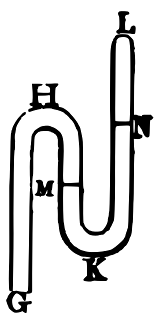
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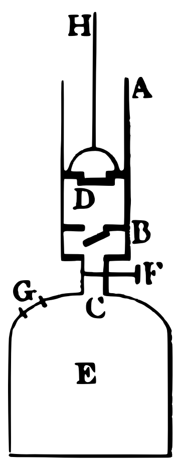
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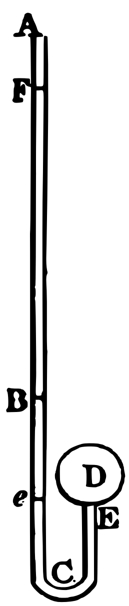
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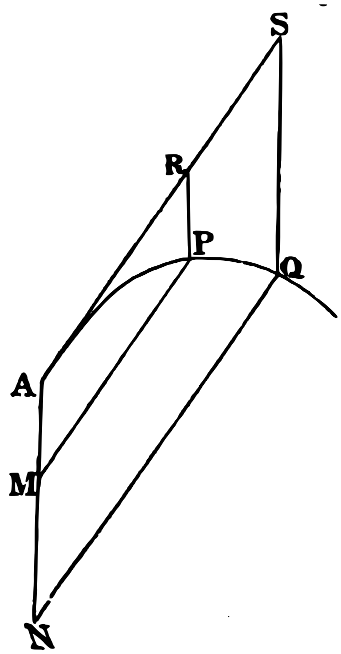
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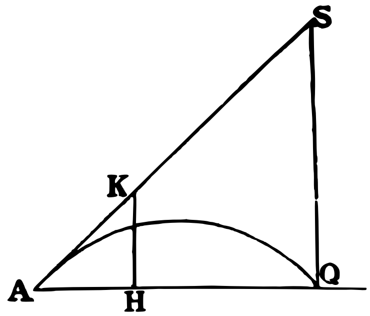
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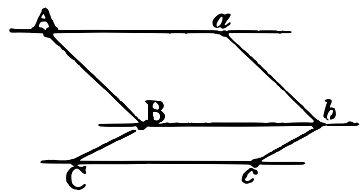
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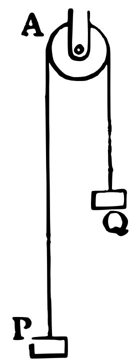
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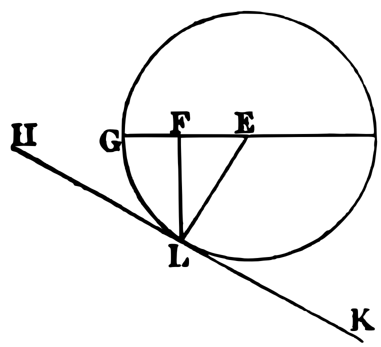
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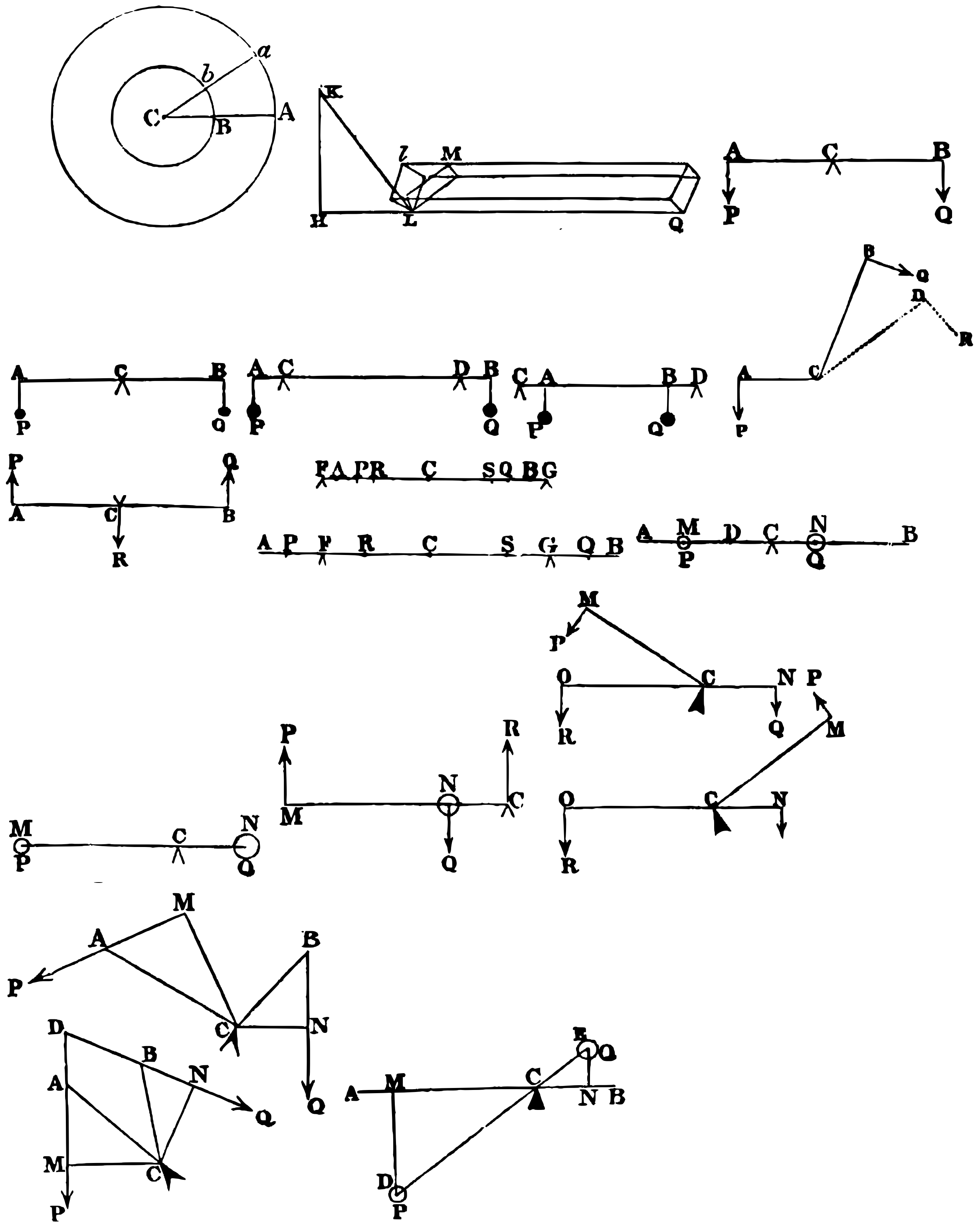

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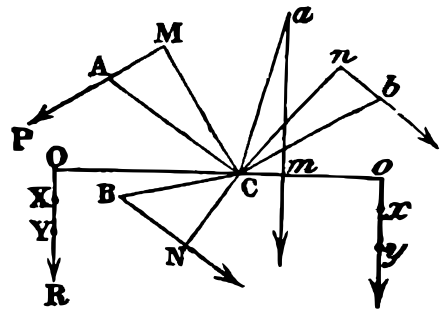
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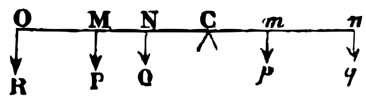
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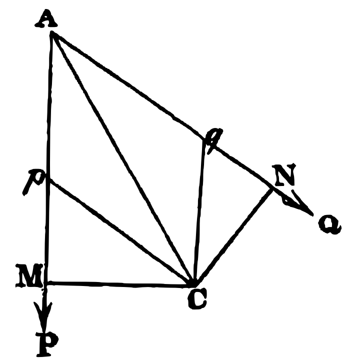
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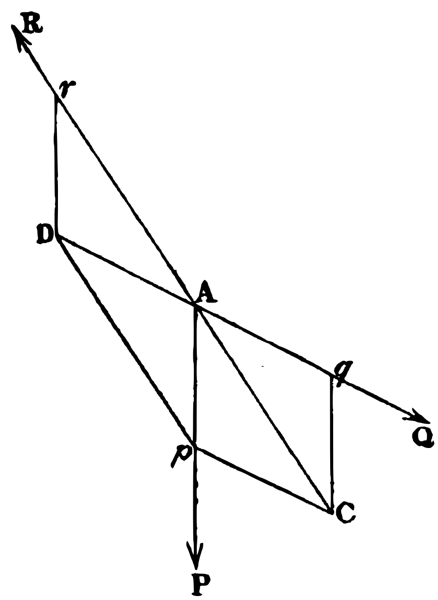
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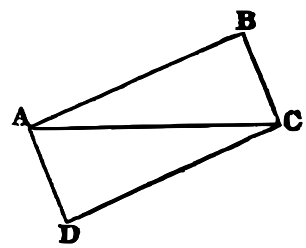
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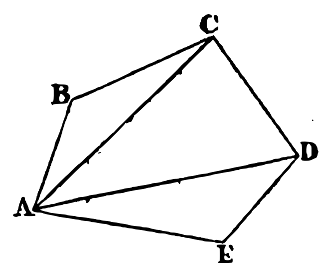
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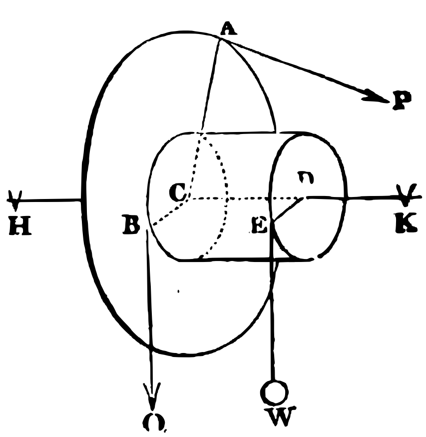
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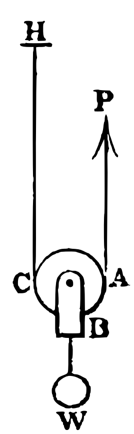
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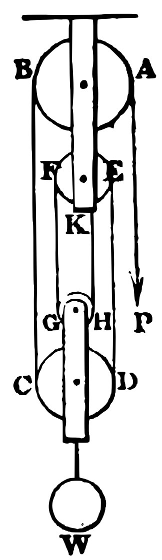
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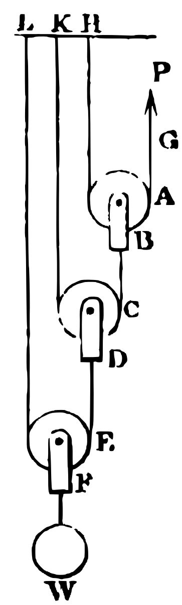
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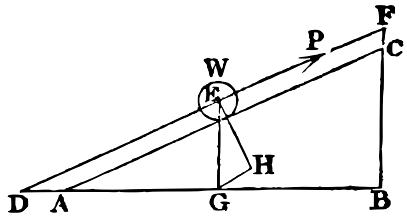
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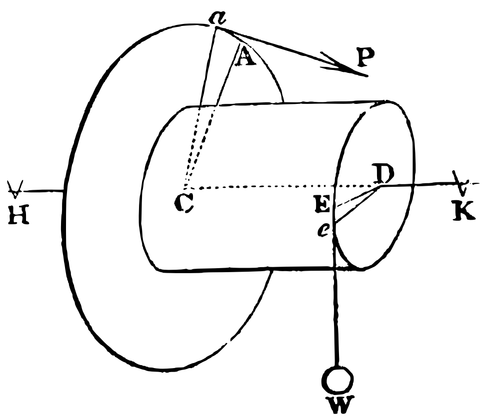
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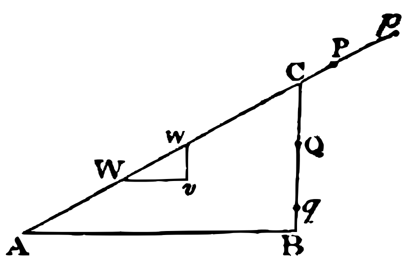
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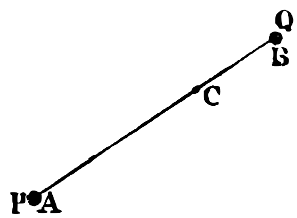
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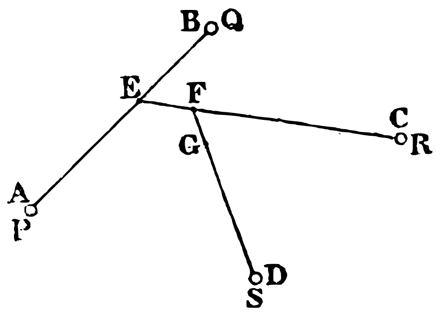
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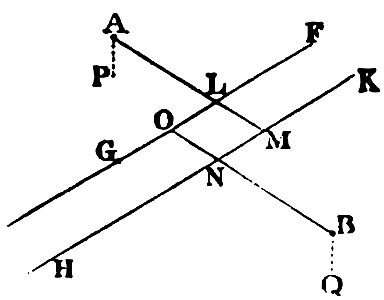
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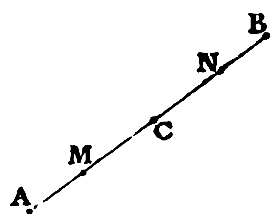
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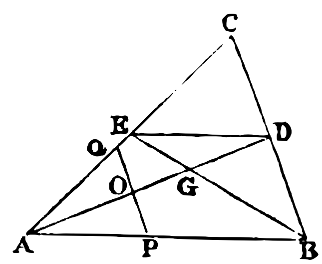
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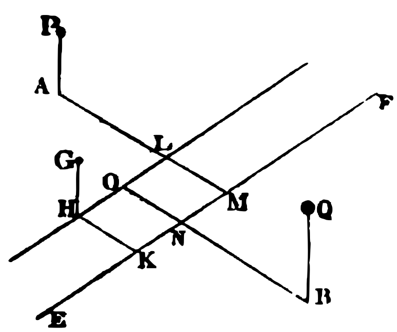
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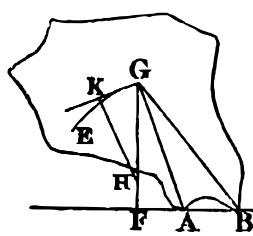
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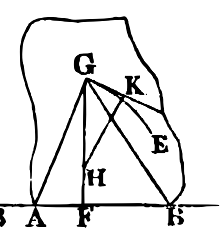
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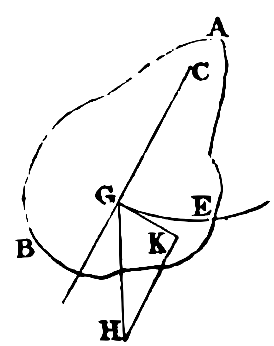
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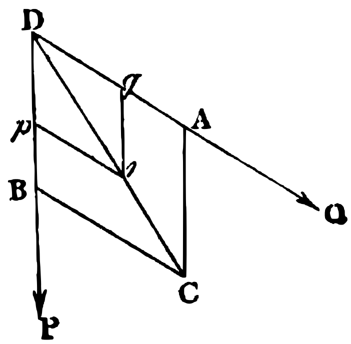
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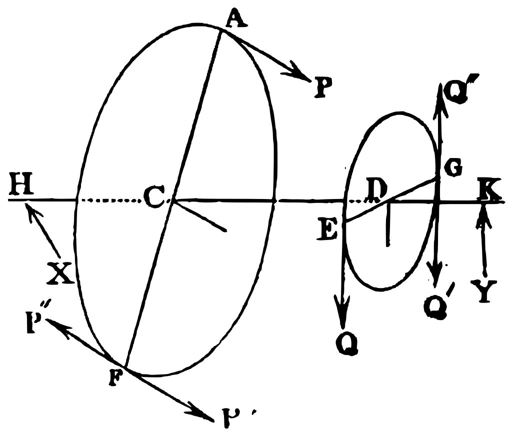
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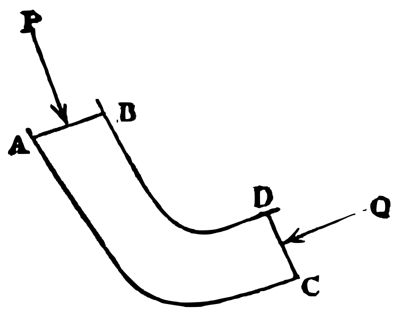
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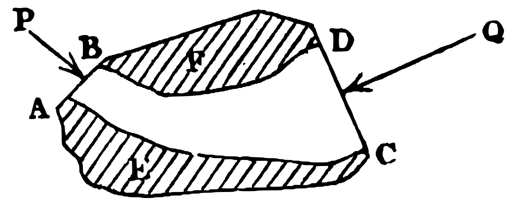
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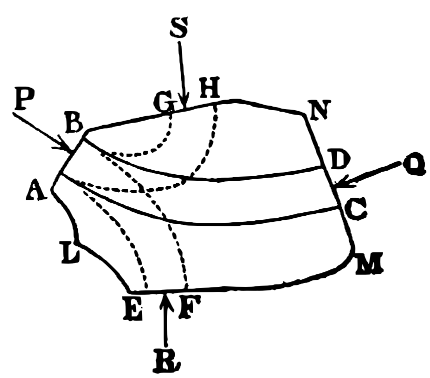
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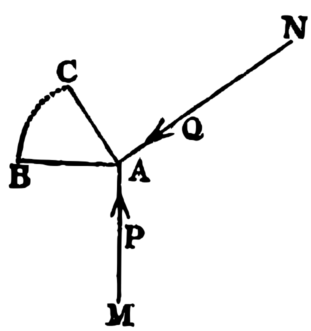
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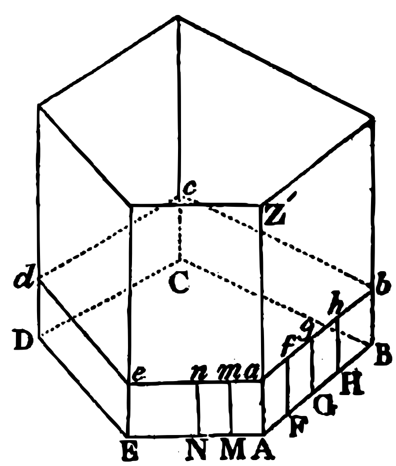
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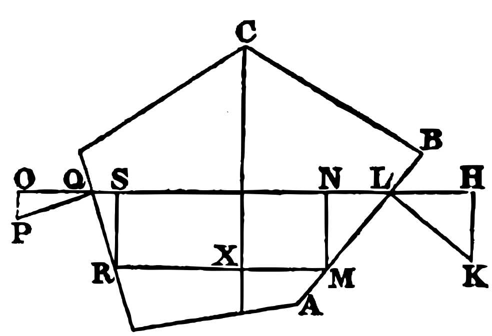
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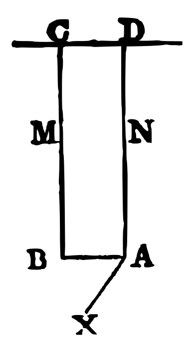
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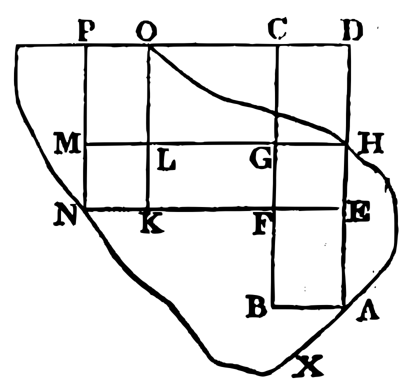
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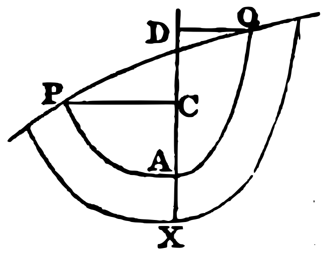
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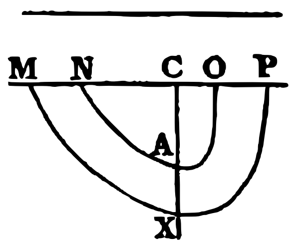
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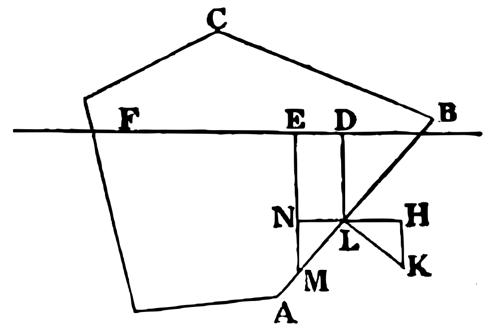
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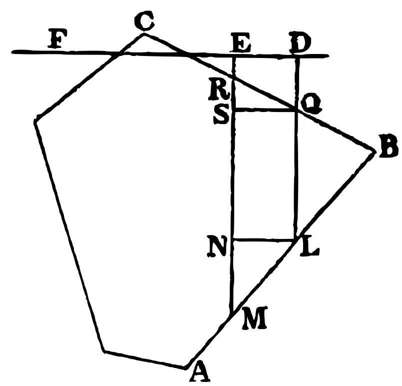
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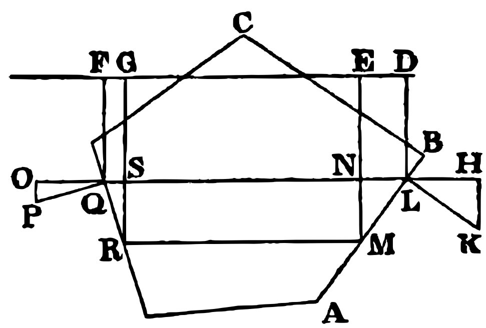
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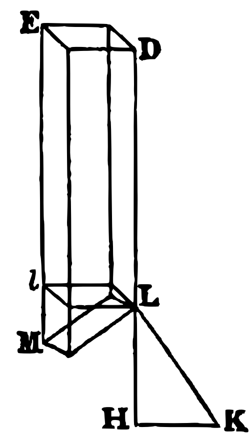
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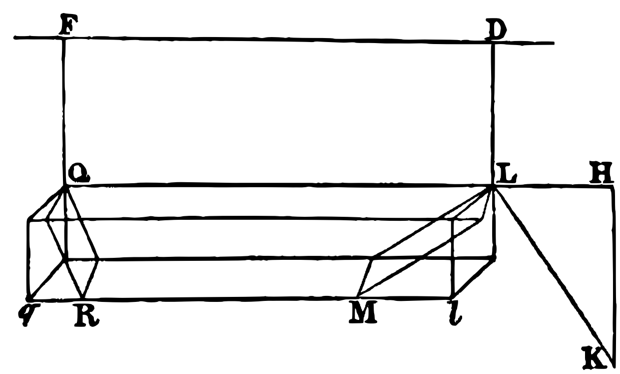
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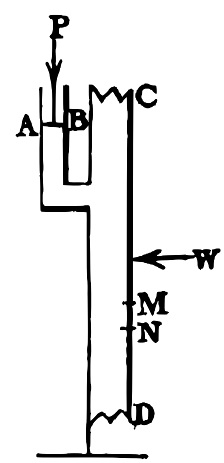
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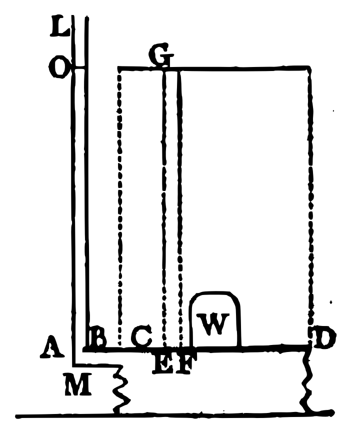
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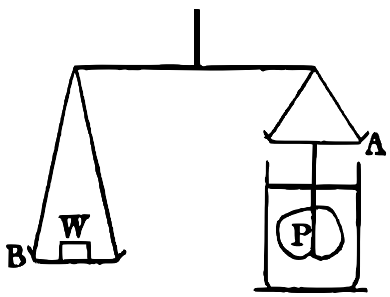

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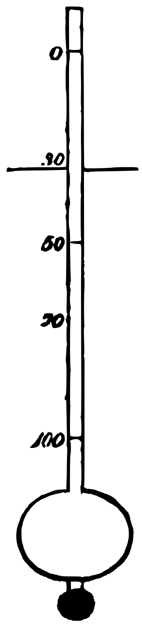

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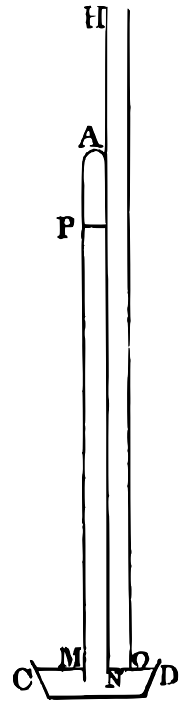

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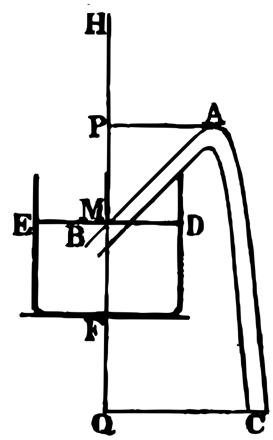

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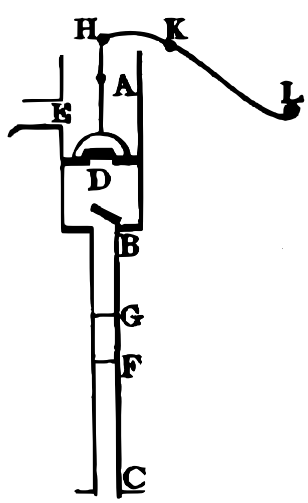

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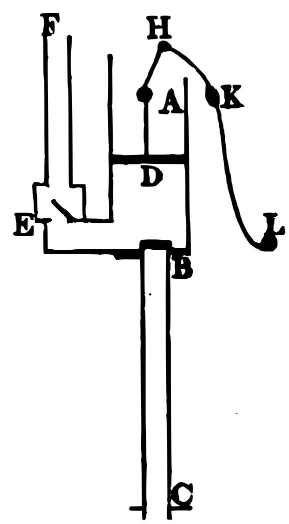

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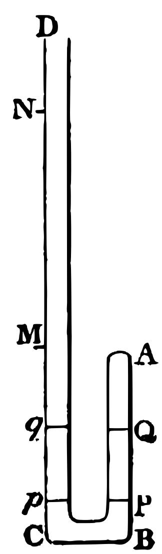

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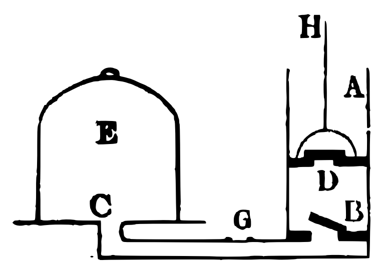

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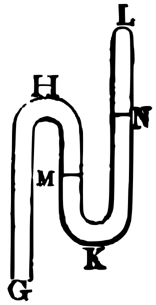

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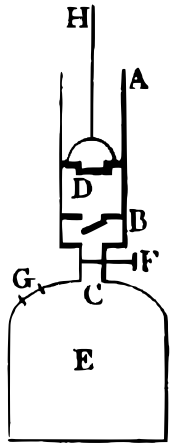

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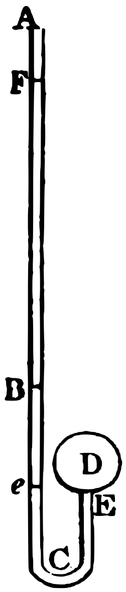

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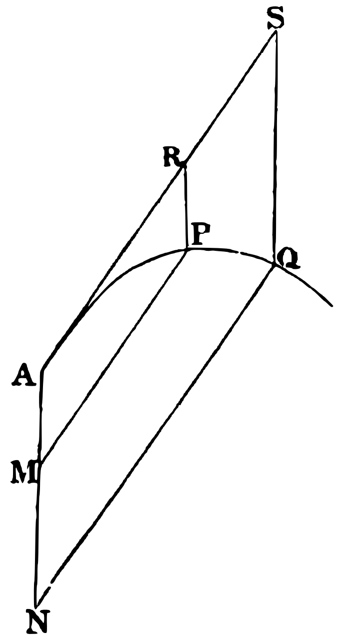

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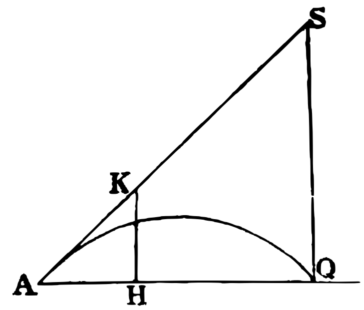

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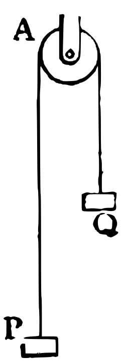

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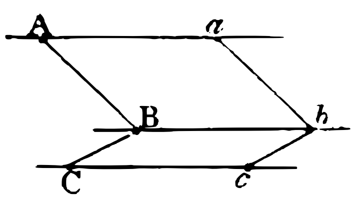

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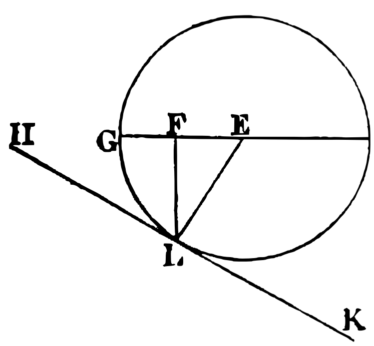

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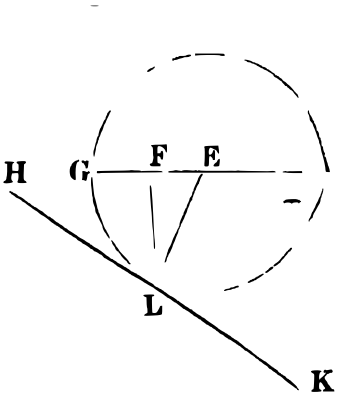

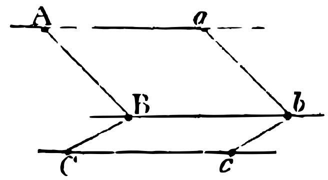

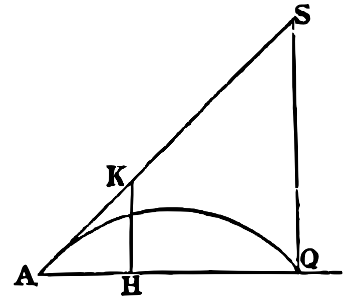

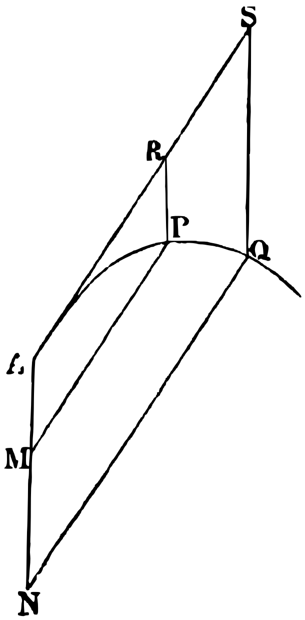

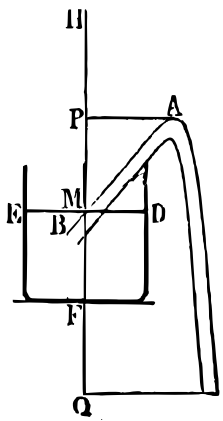

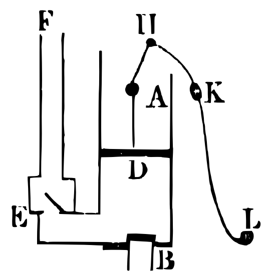

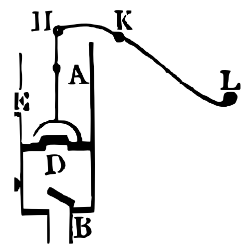

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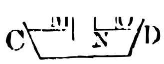
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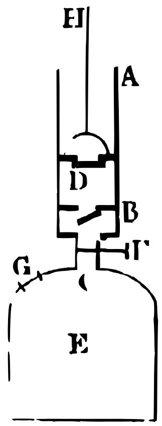
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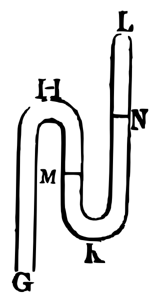
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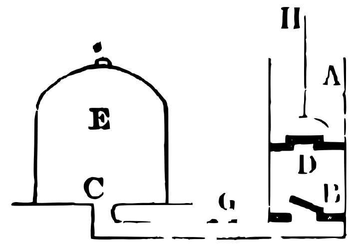
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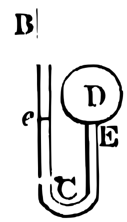
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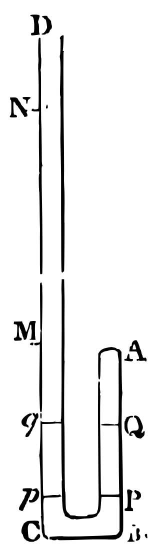
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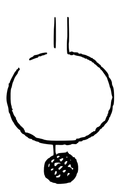

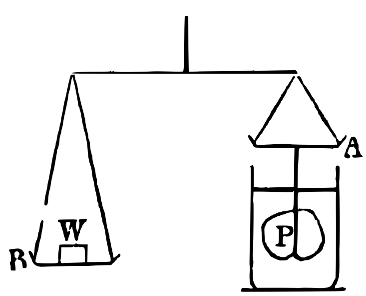
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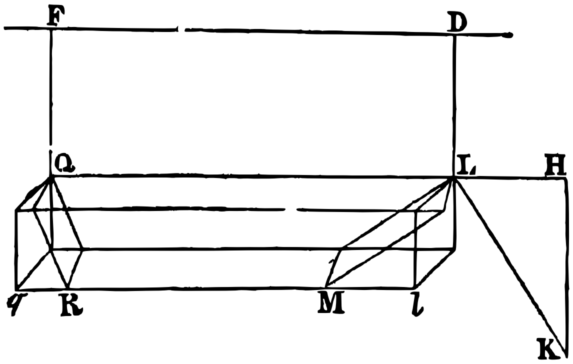
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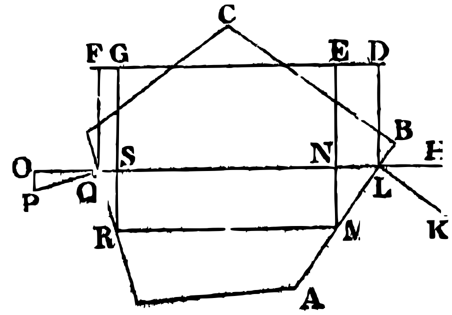
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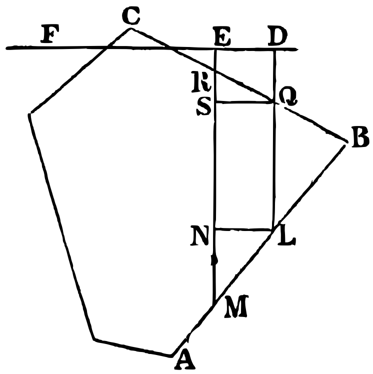
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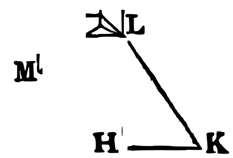
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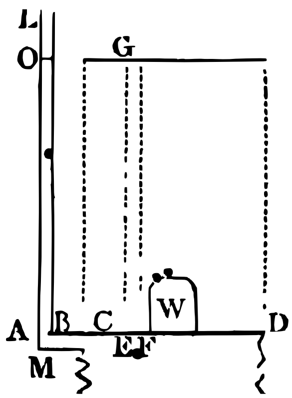
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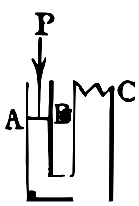
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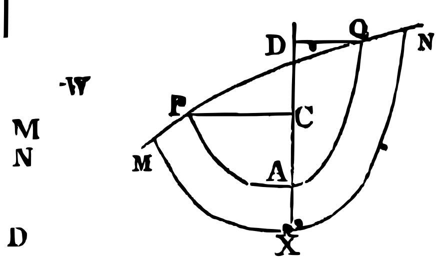
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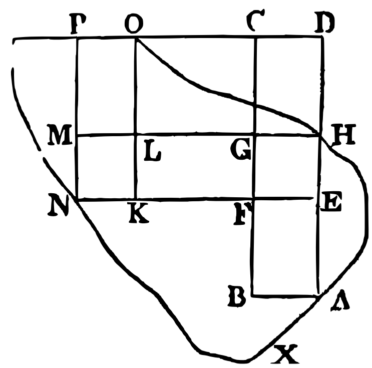
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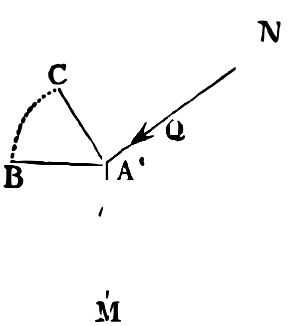
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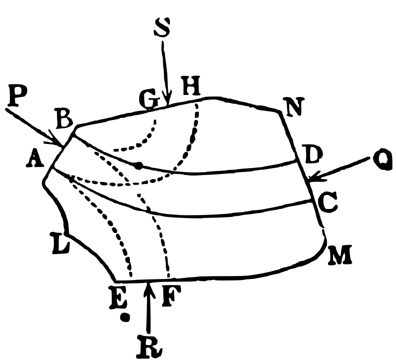
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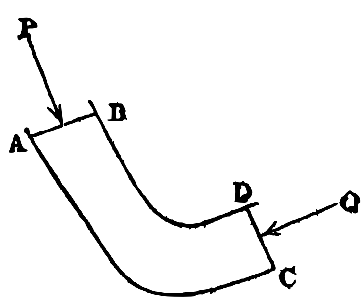
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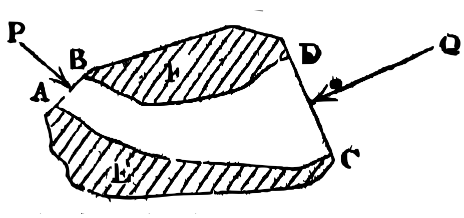
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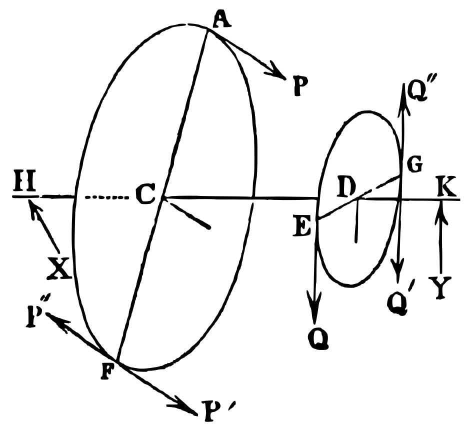
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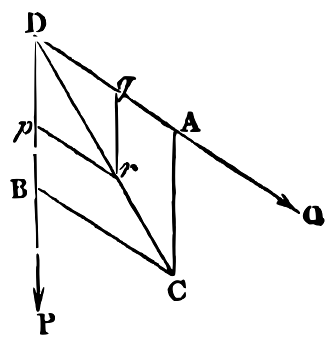
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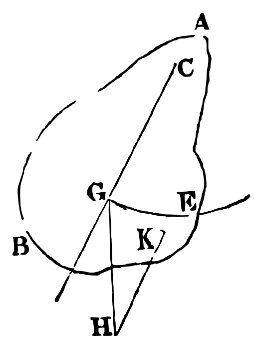
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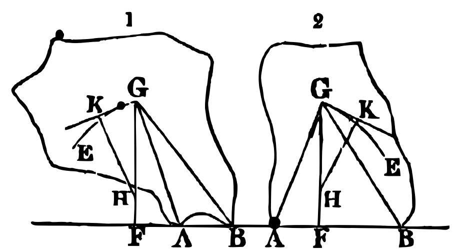
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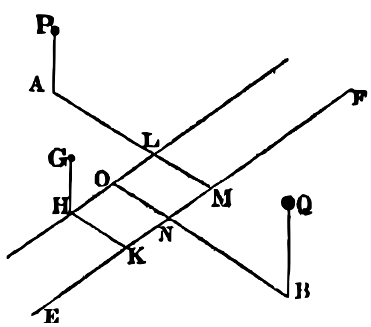
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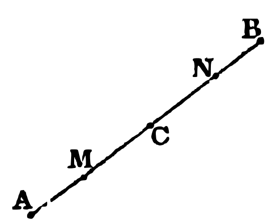
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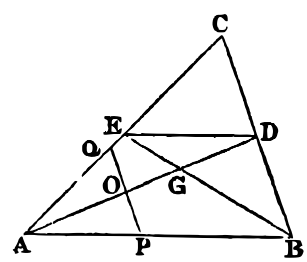
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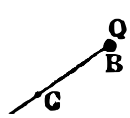
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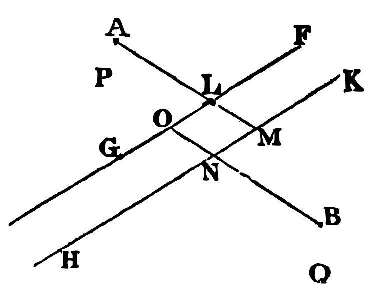
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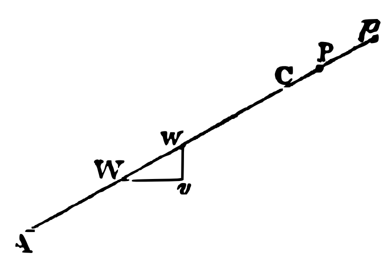
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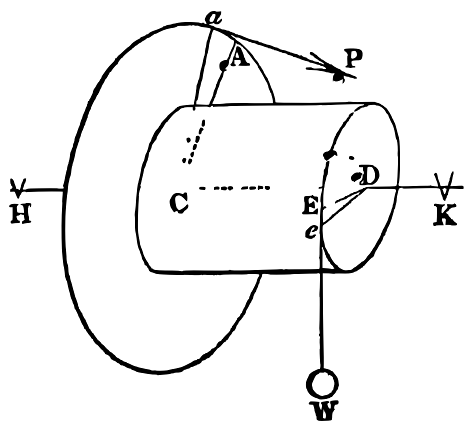
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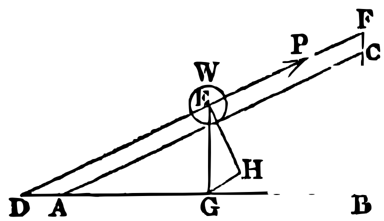
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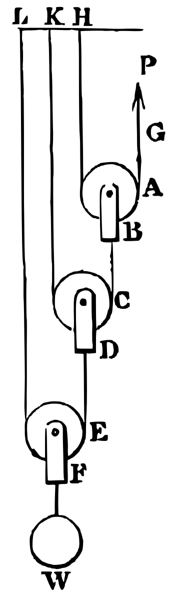
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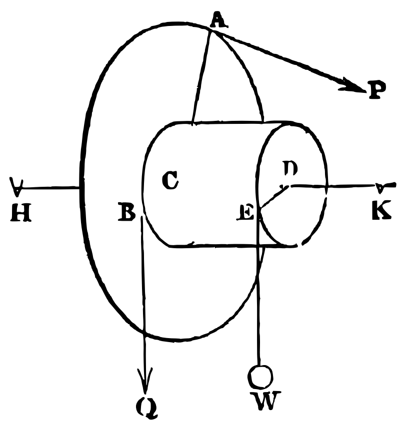
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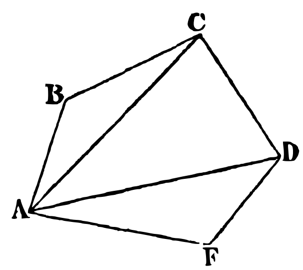
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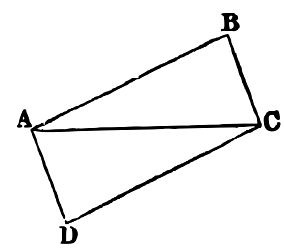
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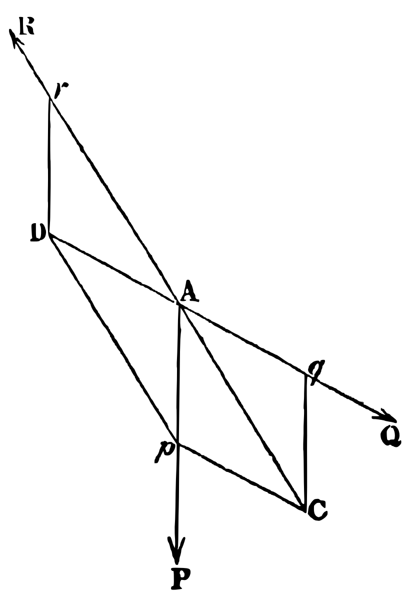
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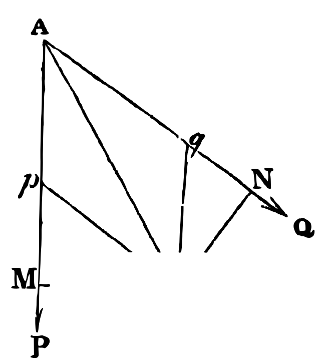
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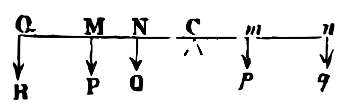
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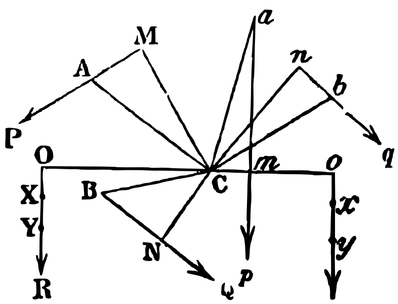

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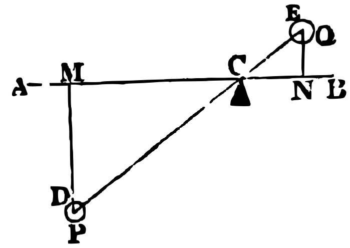

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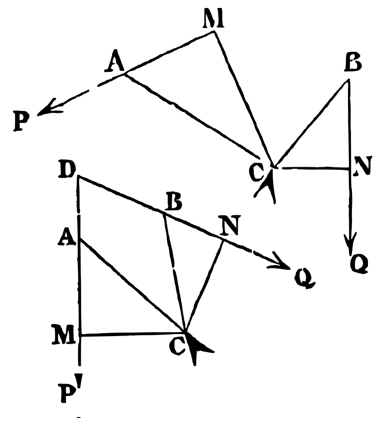

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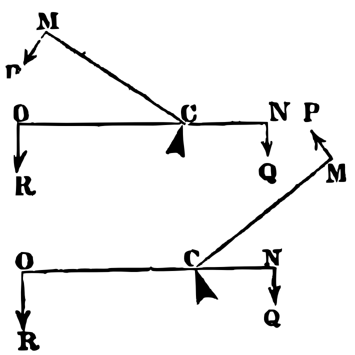

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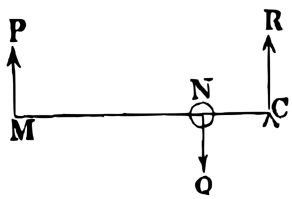

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